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Triangulated surfaces in twistor space: a kinematical set up for open/closed string duality

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Abstract: We exploit the properties of the hyperbolic space $\mathbb{H}^3$ to discuss a simplicial setting for open/closed string duality based on (random) Regge triangulations decorated with null twistorial fields. We explicitly show that the twistorial N-points function, describing Dirichlet correlations over the moduli space of open N-bordered genus g surfaces, is naturally mapped into the Witten-Kontsevich intersection theory over the moduli space of N-pointed closed Riemann surfaces of the same genus. We also discuss various aspects of the geometrical setting which connects this model to $PSL(2,\mathbb{C})$ Chern-Simons theory.

Keywords: String Duality, Lattice Models of Gravity.

Contents

1. Introduction

Major advances [1 − 9] in our understanding of Open/Closed string duality have provided a number of paradigmatical connections between Riemann moduli space theory, piecewise-linear geometry, and the study of the gauge/gravity correspondence. These connections have a two-fold origin. On the mathematical side they are deeply related to the fact that moduli space admits natural (semi-simplicial) decompositions which are in a one-to-one correspondence with classes of suitably decorated graphs. On the physical side they are consequence of the observation that these very decorated (Feynman) graphs parametrize consistently the quantum dynamics of conformal and gauge fields. In a rather general sense, simplicial techniques provide a natural kinematical framework within which we can discuss open/closed string duality. A basic problem in such a setting is to provide an explanation of how open/closed duality is dynamically generated. In particular how a closed surface is related to a corresponding open surface, with gauge-decorated boundaries, in such a way that the quantization of such a correspondence leads to a open/closed duality. Typically, the natural candidate for such a mapping is Strebel's theorem which allows to reconstruct a closed N-pointed Riemann surfaces M of genus g out of the datum of a the quadratic differential associated with a ribbon graph [10, 11] . Ribbon graphs are open Riemann surfaces which one closes by inserting punctured discs, (so generating semi-infinite cylindrical ends). The dynamics of gauge fields decorating the boundaries of the ribbon graph is naturally framed within the context of boundary conformal field theory (BCFT) which indeed plays an essential role in the onset of a open/closed duality regime. The reason for such a relevance is to be seen in the fact that BCFT is based on algebraic structures parametrized by the moduli space of genus g Riemann surfaces with N punctures $\mathcal{M}(g;N)$. This parametrization is deeply connected with Strebel's theorem in the sense that it is consistent with the operation of sewing together any two ribbon graphs (open surfaces) with (gauge-decorated) boundaries, provided that we match the complex structure and the decoration in the overlap and keep track of which puncture is ingoing and which is outgoing. In such a setting a BCFT leads to a natural algebra, over the decorated cell decomposition of Riemann moduli space, which can be related to the algebra of physical space of states of the theory and to their boundary dynamics. It is fair to say that in such a sense BCFT realizes open/closed duality as the quantization of a gauge-decorated Strebel's mapping.

It is well-known that in the analysis of the cellular geometry of Riemann moduli space there is also another point of view, pionereed by R. Penner and W. Thurston (see e.g. [12, 13]), not emphasizing the role of conformal geometry, but rather exploiting the parametrization of the moduli space in terms of hyperbolic surfaces. Here, one deals with hyperbolic surfaces with punctures (i.e. , surfaces with cuspidal geometry) rather than with surfaces with marked points. Moreover, in such a setting one generates a combinatorial decomposition of Riemann moduli space, still parametrized by ribbon graphs, not by using quadratic differentials but rather via the geometry of surface geodesics. Some geometrical aspects of the role of this particular combinatorial parametrization in open/closed string duality has been recently discussed by R. Kaufmann and R. Penner [14]. One of the advantages of the hyperbolic point of view is that one has a clear picture of the geometry of the surface. In particular of the reasons why, in assembling a surface out of ideal hyperbolic triangles, one may get an open surface with boundary, (even if the glueing pattern of the triangulation is combinatorially consistent with a closed surface) [13]. This gives a geometrical mechanism describing the transition between closed and open surfaces which, in a dynamical sense, is more interesting than Strebel's construction. The drawback is that in such a setting BCFT is not readily available and it is not obvious, (at least to us), how to formulate open/closed string duality, by, so to say, quantizing the dynamics of gauge fields on such hyperbolic combinatorial decomposition of moduli space.

The purpose of this paper is to discuss a bridge between these two combinatorial formalisms which seems to select the best of the two approaches. We use both Euclidean simplicial complexes (dual to the standard ribbon graphs) and hyperbolic geometry. This is made possible by exploiting the known correspondence existing between locally Euclidean structures in dimension two and hyperbolic geometry in dimension three. Such a correspondence allows to define in a very suggestive way a map between closed surfaces M, $\partial M = \emptyset$, triangulated with Euclidean triangles, and open hyperbolic surfaces Ω, $\partial\Omega \neq \emptyset$, triangulated by ideal hyperbolic triangles. The triangulated Euclidean closed surfaces we use are (random) Regge triangulations $|T_l| \rightarrow M$ with non-trivial curvature degrees of freedom, (represented by conical angles $\{\Theta(k)\}$, supported at its vertices, describing 2D-gravity), and decorated with null twistorial fields which can be thought of as defining the embedding of the triangulation in hyperbolic three-space $\mathbb{H}^3$. The hyperbolic surface Ω associated with such conical Euclidean triangulation is generated by locally projecting the Euclidean triangles of $|T_l| \to M$ into the 2-dimensional boundary at infinity of $\mathbb{H}^3$. By considering the upper half-space model $\mathbb{H}^{3,+}_{\text{up}}$ of $\mathbb{H}^3$, such a projection has an elementary realization in terms of the geometry of ideal tetrahedra whose vertices are decorated with (small) horospheres Σ_k. Any such a horosphere has an intrinsic Euclidean structure and can act as a projection screen Σ_∞ from which Euclidean triangles can be mapped, via hyperbolic geodesics in $\mathbb{H}^3$, into ideal hyperbolic triangles with vertices decorated by horocycles. The geometry of this geodesic projection and of the induced horocyclic decoration is quite non-trivial. The vertices $\sigma^0(k)$ of $|T_l| \to M$, with a conical defect $\Theta(k)$, get mapped into a corresponding geodesic boundary $\partial\Omega(k)$ of Ω, with a length given by $|\ln \frac{\Theta(k)}{2\pi}|$. Moreover, such boundaries come naturally endowed with a SU(2) holonomy generated by a flat su(2) connection on Ω. Null twistors naturally enter into this pictures as the consequence of the parametrization of the horospheres of $\mathbb{H}^3$ in terms of null vectors in 4-dimensional Minkowski space, (equivalently in terms of the twistorial description of the geodesics of $\mathbb{H}^3$). This correspondence between closed (singular Euclidean) surfaces and open hyperbolic surface is easily promoted to the corresponding moduli spaces: $\mathcal{M}_{g,N} \times \mathbb{R}^N_+$ the moduli spaces of N-pointed closed Riemann surfaces of genus g whose marked points are decorated with the given set of conical angles, and $\mathcal{M}_{g,N}(L)$ the moduli spaces of open Riemann surfaces of genus g with N geodesic boundaries decorated by the corresponding lengths. This provides a nice kinematical set up for establishing a open/closed string duality once the appropriate field decoration is activated. The simplest case is when we consider non-dynamical null twistors fields decorating the vertex of the Regge triangulation. These fields geometrically describe geodesics in $\mathbb{H}^3$, with an end point at $\infty \in \partial\mathbb{H}^{3,+}_{\text{up}} = (\mathbb{R}^2 \times \{0\}) \cup \{\infty\}$, projecting to the N components $\partial\Omega_k$ of the boundary of Ω. Thus, they can naturally be interpreted as fields on Ω with preassigned Dirichlet boundary conditions on the $\partial\Omega_k$'s. At the level of the moduli space $\mathcal{M}_{g,N}(L)$ we can consider the N-point function on $\mathcal{M}_{g,N}(L)$, describing correlations between such Dirichlet conditions. By exploiting a remarkable result recently obtained by Maryam Mirzakhani [15, 16], we can easily show that the such a correlation function is naturally mapped into the generating function of the Witten-Kontsevich intersection theory on $\mathcal{M}_{g,N} \times \mathbb{R}^N_+$. We can also consider the decoration of Ω associated with the $\mathfrak{su}(2)$ flat connection naturally defined on Ω (again generated by the twistorial fields, since SU(2) appears as the point stabilizer of $PSL(2,\mathbb{C})$ and one views $\mathbb{H}^3$ as the coset space $PSL(2,\mathbb{C}) \setminus \mathrm{SU}(2)$). In such a case one has a dynamic SU(2) Yang-Mills field defined on Ω and it is straightforward to explicitly write down the corresponding N-points function on $\mathcal{M}_{g,N}(L)$. The analysis of open/closed string duality in such a case is much more delicate since it also involves intersection theory over the variety $Hom(\pi_1(\Omega), \mathrm{SU}(2))/\,\mathrm{SU}(2)$ of representations, up to conjugacy, of the fundamental group of the bordered surface Ω in SU(2). Such an intersection theory is related to a careful treatment of the corresponding BCFT on Ω and calls into play Chern-Simons theory for the $PSL(2,\mathbb{C})$ group. This is still work in progress and it will be presented in a companion paper. However we thought appropriate, in a analysis mainly dealing with geometrical kinematics of open/closed duality, to conclude our presentation by discussing the aspects of the mapping between closed Regge triangulated

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surfaces $|T_l| \rightarrow M$ and open hyperbolic surfaces Ω which naturally activates $PSL(2,\mathbb{C})$ Chern-Simons theory. In simple terms the mechanism is just the parametrization of ideal tetrahedra in $\mathbb{H}^3$ in terms of the similarity structure associated with Euclidean triangles. This allows to associate with the Regge triangulated surface $|T_l| \rightarrow M$ the hyperbolic volume of a three-dimensional ideal triangulation. When such a triangulation is actually the triangulation of a hyperbolic three-manifold then the celebrated Kashaev-Murakami volume conjecture [17−21] directly activates $PSL(2,\mathbb{C})$ Chern-Simons theory.

It is also interesting to remark that Mirzakhani's results have been exploited by G. Mondello in dealing with the Poisson structure of the Teichmuller space of Riemann surfaces with boundaries [22]. This can be of relevance in describing the symplectic structure of $Hom(\pi_1(\Omega), \mathrm{SU}(2))/\,\mathrm{SU}(2)$ on such bordered surfaces and its interaction with $PSL(2,\mathbb{C})$ Chern-Simons theory.

Outline of the paper. In section 2 we sketch the properties of metrically triangulated closed surfaces and of the associated locally Euclidean (singular) structures. This is a familiar subject in simplicial quantum gravity and non-critical string theory. Here we emphasize a few delicate aspects which are not so widely known and which are relevant in our setting. In section 3 we discuss the connection between triangulated surfaces, hyperbolic three-geometry and twistors. Such a connection is fully exploited in section 4 where we explicitly construct a mapping between closed metrically triangulated surfaces and hyperbolic surfaces with boundary. We extend such a mapping to the appropriate moduli spaces and discuss an explicit example of open/closed string duality. In section 5 we analyze the connection with hyperbolic three-geometry with emphasis on the relation between the computability of the (hyperbolic) three-volume in terms of the parameters of the original two-dimensional triangulation and $PSL(2,\mathbb{C})$ Chern-Simons theory. This latter section is not really instrumental to the main body of the paper, but we have nonetheless decided to include it to further show the deep connections existing between simplicial methods and the Physics of open/closed duality. As a final comment, we would like to add a disclaimer: since some of the geometrical techniques we exploit both in the simplicial formalism as well as in hyperbolic geometry may not be widely known, we have presented a rather detailed analysis rather than qualitative arguments. As is often the case with the serious use of simplicial techniques, the notation can become at a few points quite unwieldy, and for that we apologize to the reader.

2. Random Regge triangulations

A proper understanding of the role that simplicial methods have in open/closed string duality requires that we consider metric triangulations of surfaces where both the connectivity and the edge-lengths of the triangulation are allowed to vary. Triangulations with fix connectivity but varying edge-length are known as Regge triangulations [23] wheras if we fix the edge-length and allow the connectivity to vary we get Dynamical triangulations [24, 25]. Random Regge triangulations [26], (a term being something of an oxymoron), correspond to a geometrical realization whereby both connectivity and edge-length are allowed to fluctuate and provide the general framework of our analysis.

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Let T denote an oriented finite 2-dimensional semi-simplicial complex with underlying polyhedron $|T|$, i.e. , a simplicial complex where the star $Star[\sigma^0(j)]$ of a vertex $\sigma^0(j) \in T$ (the union of all triangles of which $\sigma^0(j)$ is a face) is allowed to contain just one triangle. Denote respectively by $F(T)$, $E(T)$, and $V(T)$ the set of $N_2(T)$ faces, $N_1(T)$ edges, and $N_0(T)$ vertices of T, where $N_i(T) \in \mathbb{N}$ is the number of i-dimensional subsimplices $\sigma^i(\dots)$ $\in T$. If we assume that $|T|$ is homeomorphic to a closed surface M of genus g, then a random Regge triangulation [26] of M is a realization of the homomorphism $|T_l| \rightarrow M$ such that each edge $\sigma^1(h,j)$ of T is a rectilinear simplex of variable length $l(h,j)$. In simpler terms, T is generated by Euclidean triangles glued together by isometric identification of adjacent edges. It is important to stress that the connectivity of T is not a priori fixed as in the case of standard Regge triangulations (see [26] for details). Henceforth, if not otherwise stated, when we speak of Regge surfaces we shall always mean a Random Regge triangulated surface. We also note that in such a general setting a (semi-simplicial) dynamical triangulation $|T_{l=a}| \rightarrow M$ is a particular case [25, 24] of a random Regge PL-manifold realized by rectilinear and equilateral simplices of a fixed edge-length $l = a$.

2.1 The metric geometry of triangulated surfaces

The metric geometry of a random Regge triangulation is defined by the distribution of edge-lengths $\sigma^1(m,n) \rightarrow l(m,n)$ satisfying the appropriate triangle inequalities $l(h,j) \leq l(j,k) + l(k,h)$, whenever $\sigma^2(k,h,j) \in F(T)$. Such an assignment uniquely characterizes the Euclidean geometry of the triangles $\sigma^2(k,h,j) \in T$ and in particular, via the cosine law, the associated vertex angles $\theta_{jkh} \doteq \angle[l(j,k), l(k,h)]$, $\theta_{khj} \doteq \angle[l(k,h), l(h,j)]$, $\theta_{hjk} \doteq \angle[l(h,j), l(j,k)]$; e.g.

$$\cos\theta_{jkh} = \frac{l^2(j,k) + l^2(k,h) - l^2(h,j)}{2l(j,k)l(k,h)}\,. \tag{2.1}$$

If we note that the area $\Delta(j,k,h)$ of $\sigma^2(j,k,h)$ is provided, as a function of θ_{jkh}, by

$$\Delta(j,k,h) = \frac{1}{2} l(j,k) l(k,h)\, \sin\,\theta_{jkh}, \tag{2.2}$$

then the angles θ_{jkh}, θ_{khj}, and θ_{hjk} can be equivalently characterized by the formula

$$\cot\,\theta_{jkh} \;=\; \frac{l^2(j,k) + l^2(k,h) - l^2(h,j)}{4\,\Delta(j,k,h)}, \tag{2.3}$$

which will be useful later on. It must be stressed that the assignment

$$\begin{aligned} \mathcal{E}(T) : \left\{\sigma^2(k,h,j)\right\}_{F(T)} &\longrightarrow \mathbb{R}_+^{3\,N_2(T)} \\ \sigma^2(k,h,j) &\longmapsto (\theta_{jkh}, \theta_{khj}, \theta_{hjk}) \end{aligned} \tag{2.4}$$

of the angles θ_{jkh}, θ_{khj}, and θ_{hjk} (with the obvious constraints $\theta_{jkh} > 0$, $\theta_{khj} > 0$, $\theta_{hjk} > 0$, and $\theta_{jkh} + \theta_{khj} + \theta_{hjk} = \pi$) to each $\sigma^2(k,h,j) \in T$ does not allow to reconstruct the metric geometry of a Regge surface. $\mathcal{E}(T)$ only characterizes the local Euclidean structure [27] of $|T_l| \rightarrow M$, i.e. the similarity classes of the realization of each $\sigma^2(k,h,j)$ as an Euclidean triangle; in simpler words, their shape and not their actual size. As emphasized by Rivin [27], the knowledge of the locally Euclidean structure on $|T| \to M$ corresponds to the holonomy representation

$$H(T) : \pi_1(T \setminus V(T)) \longrightarrow GL_2(\mathbb{R}) \tag{2.5}$$

of the fundamental group of the punctured surface $M \setminus V(T)$ into the general linear group $GL_2(\mathbb{R})$, and the action of $GL_2(\mathbb{R})$ is not rigid enough for defining a coherent Euclidean glueing of the corresponding triangles $\sigma^2(k,h,j) \in T$. A few subtle properties of the geometry of Euclidean triangulations are at work here, and to put them to the fore let us consider $q(k)$ triangles $\sigma^2(k,h_\alpha,h_{\alpha+1})$ incident on the generic vertex $\sigma^0(k) \in T_l$ and generating the star

$$Star[\sigma^0(k)] \doteq \cup_{\alpha=1}^{q(k)} \sigma^2(k,h_\alpha,h_{\alpha+1}), \quad h_{q(k)+1} \equiv h_1. \tag{2.6}$$

To any given locally Euclidean structure

$$\mathcal{E}\left(Star[\sigma^0(k)]\right) \doteq \{(\theta_{\alpha+1,k,\alpha},\, \theta_{k,\alpha,\alpha+1},\, \theta_{\alpha,\alpha+1,k})\} \tag{2.7}$$

on $Star[\sigma^0(k)]$ there corresponds a conical defect $\Theta(k) \doteq \sum_{\alpha=1}^{q(k)} \theta_{\alpha+1,k,\alpha}$ supported at $\sigma^0(k)$, and a logarithmic dilatation [27], with respect to the vertex $\sigma^0(k)$, of the generic triangle $\sigma^2(k,h_\alpha,h_{\alpha+1}) \in Star[\sigma^0(k)]$, i.e. ,

$$D(k,h_\alpha,h_{\alpha+1}) \doteq \ln\sin\theta_{k,\alpha,\alpha+1} - \ln\sin\theta_{\alpha,\alpha+1,k}\,. \tag{2.8}$$

To justify this latter definition, note that if $\{l(m,n)\}$ is a distribution of edge-lengths to the triangles $\sigma^2(k,h_\alpha,h_{\alpha+1})$ of $Star[\sigma^0(k)]$ compatible with $\mathcal{E}(T)$, then by identifying $\theta_{k,\alpha,\alpha+1}$ with $\angle[l(k,h_\alpha),\, l(h_\alpha,h_{\alpha+1})]$ and $\theta_{\alpha,\alpha+1,k}$ with the angle corresponding to $\angle[\,l(h_\alpha,h_{\alpha+1}),\, l(h_{\alpha+1},k)]$, and by exploiting the law of sines, we can equivalently write

$$D(k,h_\alpha,h_{\alpha+1}) = \ln\frac{l(h_{\alpha+1},k)}{l(k,h_\alpha)}\,. \tag{2.9}$$

In terms of this parameter, we can define ([27]) the dilatation holonomy of $Star[\sigma^0(k)]$ according to

$$H(Star[\sigma^0(k)]) \doteq \sum_{\alpha=1}^{q(k)} D(k,h_\alpha,h_{\alpha+1}). \tag{2.10}$$

The vanishing of $H(Star[\sigma^0(k)])$ implies that if we circle around the vertex $\sigma^0(k)$, then the lengths $l(h_{\alpha+1},k)$ and $l(k,h_{\alpha+1})$ of the pairwise adjacent oriented edges $\sigma^1(h_{\alpha+1},k)$ and $\sigma^2(k,h_{\alpha+1})$ match up for each $\alpha = 1,\ldots,q(k)$, with $h_\alpha = h_\beta$ if $\beta = \alpha \mod q(k)$. A local Euclidean structure $\mathcal{E}(T)$ such that the dilatation holonomy $H(Star[\sigma^0(k)])$ vanishes for each choice of star $Star[\sigma^0(k)] \subset T$ is called *conically complete*. According to these remarks, the triangles in T can be coherently glued into a random Regge triangulation, with the preassigned deficit angles $\varepsilon(k) \doteq 2\pi - \Theta(k)$ generated by the given $\mathcal{E}(T)$, if and only if $\mathcal{E}(T)$ is complete in the above sense. Note that if the deficit angles $\{\varepsilon(k)\}_{V(T)}$ all vanish, we end up in the more familiar notion of holonomy associated with the completeness of the Euclidean structure associated with $|T| \to M$ and described by a developing map whose rotational holonomy around any vertex is trivial.

2.2 Holonomies of singular Euclidean structures

There is a natural way to keep track of both dilation factors and conical defects by introducing a complex-valued holonomy. Let $(\zeta_k, \zeta_h, \zeta_j)$ a ordered triple of complex numbers describing the vertices of a realization, in the complex plane $\mathbb{C}$, of the oriented triangle $\sigma^2(k,h,j)$, with edge lenghts $l(k,h)$, $l(h,j)$, $l(j,k)$. By using Euclidean similarities we can always map $(\zeta_k, \zeta_h, \zeta_j)$ to $(0,1,\zeta_{jkh})$, with

$$\zeta_{jkh} \doteq \frac{\zeta_j - \zeta_k}{\zeta_h - \zeta_k} = \frac{l(j,k)}{l(k,h)}\, e^{i\,\theta_{jkh}}, \tag{2.11}$$

where

$$\arg\, \zeta_{jkh} = \arg\,(\zeta_j - \zeta_k) - \arg\,(\zeta_h - \zeta_k) = \theta_{jkh} \in [0, 2\pi), \tag{2.12}$$

(thus Im $\zeta_{jkh} > 0$). The triangle $(0,1,\zeta_{jkh})$ is in the same similarity class $\mathcal{E}(\sigma^2(k,h,j))$ of $\sigma^2(k,h,j)$, and the vector ζ_{jkh}, the *complex modulus* of the triangle $\sigma^2(k,h,j)$ with respect to $\sigma^0(k)$, parametrizes $\mathcal{E}(\sigma^2(k,h,j))$. The same similarity class is obtained by cyclically permuting the vertex which is mapped to 0, i.e. , $(\zeta_h, \zeta_j, \zeta_k) \to (0,1,\zeta_{khj})$ and $(\zeta_j, \zeta_k, \zeta_h) \to (0,1,\zeta_{hjk})$, where

$$\zeta_{khj} \doteq \frac{\zeta_k - \zeta_h}{\zeta_j - \zeta_h} = \frac{l(k,h)}{l(h,j)}\, e^{i\,\theta_{khj}}, \tag{2.13}$$

$$\zeta_{hjk} \doteq \frac{\zeta_h - \zeta_j}{\zeta_k - \zeta_j} = \frac{l(h,j)}{l(j,k)}\, e^{i\,\theta_{hjk}}, \tag{2.14}$$

are the moduli of $\mathcal{E}(\sigma^2(k,h,j))$ with respect to the vertex $\sigma^0(h)$ and $\sigma^0(j)$, respectively. Elementary geometrical considerations imply that the triangles $(\zeta_{jkh}\,\zeta_{hjk}\,\zeta_{khj}, 0, \zeta_{hjk}\,\zeta_{jkh})$, $(0,1,\zeta_{jkh})$, and $(\zeta_{jkh}, \zeta_{hjk}\,\zeta_{jkh}, 0)$ are congruent. This yields the relations

$$\begin{aligned} \zeta_{jkh}\,\zeta_{hjk}\,\zeta_{khj} &= -1, \\ \zeta_{jkh}\,\zeta_{hjk} &= \zeta_{jkh} - 1, \end{aligned} \tag{2.15}$$

according to which a choice of a moduli with respect a particular vertex specifies also the remaining two moduli. For instance, if we describe $\mathcal{E}(\sigma^2(k,h,j))$ by the modulus $\zeta_{jkh} \doteq \zeta$, (Im $\zeta > 0$), with respect to $\sigma^0(k)$ then we get

$$\begin{aligned} \zeta_{jkh} &\doteq \zeta, \\ \zeta_{khj} &= \frac{1}{1-\zeta}, \\ \zeta_{hjk} &= 1 - \frac{1}{\zeta}. \end{aligned} \tag{2.16}$$

By selecting the standard branch on $\mathbb{C} - (-\infty, 0]$ of the natural logarithm, we also get

$$\begin{aligned} \ln \zeta_{jkh} &= \ln \zeta, \\ \ln \zeta_{khj} &= -\ln(1-\zeta), \\ \ln \zeta_{hjk} &= \ln(1-\zeta) - \ln\zeta + \pi\, i. \end{aligned} \tag{2.17}$$

In terms of these log-parameters we can extend the logarithmic dilation of the generic triangle $\sigma^2(k, h_\alpha, h_{\alpha+1}) \in Star[\sigma^0(k)]$, to its complexified form

$$D^{\mathbb{C}}(k, h_\alpha, h_{\alpha+1}) \doteq \ln \zeta_{h_{\alpha+1},k,h_\alpha} = \ln \frac{l(h_{\alpha+1}, k)}{l(k, h_\alpha)} + i\ \theta_{h_{\alpha+1},k,h_\alpha}, \tag{2.18}$$

where $\zeta_{h_{\alpha+1},k,h_\alpha}$ is the complex modulus of the triangle $\sigma^2(k, h_\alpha, h_{\alpha+1})$ with respect to the vertex $\sigma^0(k)$. Correspondingly we define

$$\begin{aligned} H^{\mathbb{C}}(Star[\sigma^0(k)]) &\doteq \sum_{\alpha=1}^{q(k)} D^{\mathbb{C}}(k, h_\alpha, h_{\alpha+1}) \qquad (2.19)\\ &= \sum_{\alpha=1}^{q(k)} D(k, h_\alpha, h_{\alpha+1}) + i \sum_{\alpha=1}^{q(k)} \theta_{h_{\alpha+1},k,h_\alpha} = H(Star[\sigma^0(k)]) + i\ \Theta(k), \end{aligned}$$

where $\Theta(k)$ is the conical defect supported at the vertex $\sigma^0(k)$. According to the previous remarks, it follows that the triangulation $|T_l| \to M$ will be conicaly complete iff Re $H^{\mathbb{C}}(Star[\sigma^0(k)]) = 0$ for every vertex star, and its conical defects are provided by $\{\text{Im } H^{\mathbb{C}}(Star[\sigma^0(k)])\}$. In other words, a necessary and sufficient condition on the locally Euclidean structure $\{\theta_{jkh}, \theta_{khj}, \theta_{hjk}\}_{F(T)}$ in order to define a glueing and hence a random Regge triangulation is the requirement that

$$\prod_{k=1}^{q(k)} \zeta_{h_{\alpha+1},k,h_\alpha} \in \mathrm{U}(1), \tag{2.20}$$

for each $Star[\sigma^0(k)]$, i.e. that the image of $H^{\mathbb{C}}(T)$ lies in the group $\mathrm{U}(1)$. Note that the condition for having a flat Regge triangulation is stronger than $\prod_{k=1}^{q(k)} \zeta_{h_{\alpha+1},k,h_\alpha} = 1$, $\forall k = 1, \ldots, N_0(T)$, since it requires that

$$\sum_{k=1}^{N_0(T)} H^{\mathbb{C}}(Star[\sigma^0(k)]) = 2N_0(T)\pi\, i. \tag{2.21}$$

3. Regge surfaces and ideal tetrahedra in $\mathbb{H}^3$

The connection between similarity classes of arrangements of Euclidean triangles with trivial holonomy $H^{\mathbb{C}}$ and triangulations of three-manifolds by ideal tetrahedra is a well-known property of three-dimensional hyperbolic geometry [28]. This interplay extends in a subtle way to the case in which $H^{\mathbb{C}}$ is no longer trivial (i.e. , to singular Euclidean structure [27]) and plays a key role in our results.

To set the stage, let $\mathbb{H}^3$ denote the 3-dimensional hyperbolic space thought of as the subspace of Minkowski spacetime $(M^4, \langle\cdot,\cdot\rangle)$ defined by

$$\mathbb{H}^3 = \left\{ \overrightarrow{x} \doteq \left(x^0, x^1, x^2, x^3\right) \mid \langle \overrightarrow{x}, \overrightarrow{x} \rangle = -1,\ x^0 > 0 \right\}, \tag{3.1}$$

and equipped with the induced Riemannian metric defined by the restriction to the tangent spaces $T_x\mathbb{H}^3$ of the standard Minkowski inner product

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle \doteq -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3. \tag{3.2}$$

Recall that the group of orientation preserving isometries of $\mathbb{H}^3$ can be identified with the group $PSL(2,\mathbb{C})$ which acts transitively on $\mathbb{H}^3$ with point stabilizer provided by SU(2). Let $x \in \mathbb{H}^3$ and $\overrightarrow{y} \in T_x\mathbb{H}^3$ with $\langle \overrightarrow{y}, \overrightarrow{y} \rangle = 1$, then the geodesic in $\mathbb{H}^3$ starting at x with velocity $\overrightarrow{y}$ is traced by the intersection of $\mathbb{H}^3$ with the two-dimensional hyperplane of M^4 generated by the position vector $\overrightarrow{x}$ and the velocity $\overrightarrow{y}$ and is described by the mapping

$$\mathbb{R} \ni t \longmapsto \gamma(t) = \cosh(t)\, \overrightarrow{x} + \sinh(t)\, \overrightarrow{y}. \tag{3.3}$$

Let $\gamma(\infty)$ denote the endpoint of γ on the sphere at infinity $\partial\mathbb{H}^3 \simeq \mathbb{S}^2$, a closed horosphere centered at $\gamma(\infty)$ is a closed surface $\Sigma \subset \mathbb{H}^3$ which is orthogonal to all geodesic lines in $\mathbb{H}^3$ with endpoint $\gamma(\infty)$. The horospheres in $\mathbb{H}^3$ with centres at $\gamma(\infty)$ can be defined as the level set of the Busemann function $\varphi : \mathbb{H}^3 \to \mathbb{R}$ associated with γ and defined by

$$\varphi(x) \doteq \lim_{t\to\infty} d_{\mathbb{H}^3}(x, \gamma(t)) - t, \tag{3.4}$$

where $d_{\mathbb{H}^3}(\ ,\ )$ denotes the hyperbolic distance. Thus, to each points at infinity $\gamma(\infty) \in \partial\mathbb{H}^3$ is associated a foliation of $\mathbb{H}^3$ by horospheres which are the level sets of the Busemann function. In particular, two horospheres with centre at the same point at infinity are at a constant distance. Note that, as a set, the horospheres can be parametrized by future-pointing null vectors belonging to the future light-cone

$$\mathbb{L}^+ \doteq \left\{ \overrightarrow{x} \doteq \left(x^0, x^1, x^2, x^3\right) \mid \langle \overrightarrow{x}, \overrightarrow{x} \rangle = 0,\ x^0 > 0 \right\} \tag{3.5}$$

by identifying the generic horosphere Σ_w with the intersection between $\mathbb{H}^3$ and the null hyperplane $\langle \overrightarrow{y}, \overrightarrow{w} \rangle = -1$ defined by the null vector $\overrightarrow{w}$, i.e. ,

$$\overrightarrow{w} \longmapsto \Sigma_w \doteq \left\{ y \in \mathbb{H}^3 \mid \langle \overrightarrow{y}, \overrightarrow{w} \rangle = -1,\ \langle \overrightarrow{w}, \overrightarrow{w} \rangle = 0 \right\}. \tag{3.6}$$

Such an identification allows to associate a natural functional with any pair of horospheres Σ_u and Σ_v according to

$$\lambda\left(\Sigma_u, \Sigma_v\right) \doteq \sqrt{-\langle \overrightarrow{u}, \overrightarrow{v} \rangle}. \tag{3.7}$$

the quantity $\lambda\left(\Sigma_u, \Sigma_v\right)$ defines the *lambda length* [12] between Σ_u and Σ_v. If $\gamma(p,q)$ denotes the unique geodesic in $\mathbb{H}^3$ connecting the respective centers p and q of Σ_u and Σ_v, then $\lambda\left(\Sigma_u, \Sigma_v\right)$ can be related do the signed geodesic distance $\delta(u,v)$ between the intersection points $\gamma(p,q) \cap \Sigma_u$ and $\gamma(p,q) \cap \Sigma_v$, according to

$$\lambda\left(\Sigma_u, \Sigma_v\right) = \sqrt{2\, e^{\delta(u,v)}}, \tag{3.8}$$

($\delta(u,v)$ is by convention < 0 if Σ_u and Σ_v cross each other).

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In order to discuss the connection between Regge triangulations and hyperbolic geometry, it will be convenient to represent $\mathbb{H}^3$ by the upper half-space model $\mathbb{H}^{3,+}_{\text{up}}$, i.e. as the open upper half space $\{(X,Y,Z)\in\mathbb{R}^3 \mid Z>0\}$ endowed with the Poincaré metric $Z^{-2}(dX^2+dY^2+dZ^2)$. The boundary of $\mathbb{H}^3$ is here provided by $\partial\mathbb{H}^{3,+}_{\text{up}}=(\mathbb{R}^2\times\{0\})\cup\{\infty\}$, and, up to isometries, we can always map a given point p to ∞. Geodesics in the half-space model are obtained by parametrization of vertical lines $\{x\}\times\mathbb{R}_+$ and circles orthogonal to $\mathbb{R}^2\times\{0\}$. In particular, since geodesics with end point ∞ are vertical lines, it easily follows that in $\mathbb{H}^{3,+}_{\text{up}}$ the horospheres (centered at ∞) are horizontal hyperplanes. It is also worthwhile recalling that the hyperbolic distance between two points p, and $q\in\mathbb{H}^3$ is explicitly provided in $\mathbb{H}^{3,+}_{\text{up}}$ by

$$d_{\mathbb{H}^3}(p,q)=2\tanh^{-1}\left[\frac{(X_p-X_q)^2+(Y_p-Y_q)^2+(Z_p-Z_q)^2}{(X_p-X_q)^2+(Y_p-Y_q)^2+(Z_p+Z_q)^2}\right]^{\frac{1}{2}}. \tag{3.9}$$

In particular, if we take any two geodesics l_1 and l_2 with end-point ∞ and evaluate their hyperbolic distance $d_{\mathbb{H}^3}(l_1,l_2)$ along the horospheres $\Sigma_1\doteq\{z=t_1\}$ and $\Sigma_2\doteq\{z=t_2\}$, with $t_2>t_1$, separated by a distance $d_{\mathbb{H}^3}(\Sigma_1,\Sigma_2)$, then we get the useful relation

$$d_{\mathbb{H}^3}(l_1,l_2)|_{\Sigma_2}=d_{\mathbb{H}^3}(l_1,l_2)|_{\Sigma_1}\,e^{d_{\mathbb{H}^3}(\Sigma_1,\Sigma_2)}. \tag{3.10}$$

Let $\sigma^3_{\text{hyp}}\doteq(v^0(0),v^0(k),v^0(h),v^0(j))$ be an ideal simplex in $\mathbb{H}^{3,+}_{\text{up}}$, i.e. , a simplex whose faces are hyperbolic triangles, edges are geodesics, and with vertices lying on $\partial\mathbb{H}^{3,+}_{\text{up}}$. In order to describe the basic properties of σ^3_{hyp} recall that, up to isometries of $\mathbb{H}^{3,+}_{\text{up}}$, we can always assume that one of its four vertices, say $v^0(0)$, is at the point ∞ whereas the remaining three $v^0(k)$, $v^0(h)$, and $v^0(j)$ lie on the circumference intersection of $\mathbb{R}^2\times\{0\}$ with a Euclidean half-sphere $\mathbb{D}^2_r$ of radius r and centre $c\in\{(X,Y,Z)\in\mathbb{R}^3\,|\,Z=0\}$. Note that $\mathbb{D}^2_r$ inherits from $\mathbb{H}^{3,+}_{\text{up}}$ the structure of a two-dimensional hyperbolic space and that, consequently the simplex $\sigma^2_{\text{hyp}}\doteq(v^0(k),v^0(h),v^0(j))$, providing the two-dimensional face of $\sigma^3_{\text{hyp}}\doteq(v^0(0),v^0(k),v^0(h),v^0(j))$ opposite to the vertex $v^0(0)\simeq\infty$, is itself an ideal simplex in $\mathbb{D}^2_r$. Denote by $\Delta_\infty(v^0(0))$ the intersection between σ^3_{hyp} and a horosphere Σ_∞ centered at $v^0(0)\doteq\infty$ and sufficiently near to $v^0(0)$. Since all horospheres are congruent, Σ_∞ can be mapped onto a horizontal plane $z=t\subset\mathbb{H}^{3,+}_{\text{up}}$ by a conformal mapping fixing ∞, to the effect that $\Delta_\infty(v^0(0))$ is a Euclidean triangle $T_\infty(\sigma^3_{\text{hyp}})\equiv\sigma^2(k,h,j)$ in the plane of the horosphere. This latter remark implies that the vertex angles $(\theta_{jkh},\theta_{khj},\theta_{hjk})$ of $T_\infty(\sigma^3_{\text{hyp}})$ can be identified with the inner dihedral angles at the three edges $v^1(\infty,k)$, $v^1(\infty,h)$, and $v^1(\infty,j)$ of σ^3_{hyp}, i.e. ,

$$\begin{aligned}\theta_{jkh}&\longmapsto\phi_{\infty k}\doteq\angle\left[v^2(0,j,k),v^2(0,k,h)\right],\\ \theta_{khj}&\longmapsto\phi_{\infty h}\doteq\angle\left[v^2(0,k,h),v^2(0,h,j)\right],\\ \theta_{hjk}&\longmapsto\phi_{\infty j}\doteq\angle\left[v^2(0,h,j),v^2(0,j,k)\right],\end{aligned} \tag{3.11}$$

where $v^2(.,.,.)$ denote the faces of σ^3_{hyp}. It is easy to prove, again by intersecting σ^3_{hyp} with horospheres Σ_k, Σ_h, Σ_j centered and sufficiently near to the respective vertices $v^0(k)$, $v^0(h)$, $v^0(j)$, that dihedral angles along opposite edges in σ^3_{hyp} are pairwise equal $\phi_{\infty k}=\phi_{hj}$,

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$\phi_{\infty h} = \phi_{jk}$, $\phi_{\infty j} = \phi_{kh}$. This implies that the (Euclidean) triangles cut by the horospheres Σ_k, Σ_h, Σ_j are all similar to $T_\infty(\sigma^3_{\rm hyp})$. In particular, note that the geometrical realizations of the simplices

$$\begin{aligned} \sigma^2_{\rm hyp}(k,h,j) &\doteq v^2(k,h,j), \\ \sigma^2_{\rm hyp}(\infty,k,h) &\doteq v^2(0,k,h), \\ \sigma^2_{\rm hyp}(\infty,j,k) &\doteq v^2(0,j,k), \end{aligned} \tag{3.12}$$

are ideal triangles in $\mathbb{H}^3$. It follows that the above construction is independent from the choice of which of the four vertices of $\sigma^3_{\rm hyp}$ is mapped to ∞ and we can parametrize the ideal tetrahedra $\sigma^3_{\rm hyp}$ in $\mathbb{H}^{3,+}_{up}$ in terms of the similarity class $[\sigma^2(k,h,j)]$ of the associated Euclidean triangle $T(\sigma^3_{\rm hyp})$: any two ideal tetrahedra $\sigma^3_{\rm hyp}$ in $\mathbb{H}^{3,+}_{\rm up}$ are congruent iff the associated triangles $T(\sigma^3_{\rm hyp})$ are similar. This is in line with the basic property of $\mathbb{H}^3$ according to which if a diffeomorphism of $\mathbb{H}^3$ preserves angles then it also preserves lengths.

3.1 Horospheres and twistors

To conclude this brief résumé of hyperbolic geometry, let us observe that if we mark a point P on the horosphere Σ_w, then there is a characterization of (Σ_w, P) in terms of *null twistors* which will be relevant in what follows. To fix notation, let $\sigma_0^{AA'} \doteq \delta^{AA'}$ and denote by

$$\sigma_1^{AA'} \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2^{AA'} \doteq \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3^{AA'} \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.13}$$

the Pauli matrices. Let l_w the null line passing through the marked point $P \in \Sigma_w$ and contained in the plane $\langle \overrightarrow{y}, \overrightarrow{w} \rangle = -1$ whose intersection with $\mathbb{H}^3$ defines Σ_w. Let us represent the coordinates y^k of P in terms of the Hermitian matrix

$$P_w \longmapsto y^{AA'} = \frac{1}{\sqrt{2}} y^k \sigma_k^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} y^0 + y^3 & y^1 + i\, y^2 \\ y^1 - i\, y^2 & y^0 - y^3 \end{pmatrix} \tag{3.14}$$

and describe the null future-pointing vector $\overrightarrow{w} \in \mathbb{L}^+$ in terms of a two components $\mathrm{SL}(2,\mathbb{C})$ spinor ξ^A as $\overrightarrow{w} \longleftrightarrow \xi^A \overline{\xi}^{A'}$, (as usual, a representation which is unique up to a phase conjugation, i.e., $\xi^A \mapsto e^{i\varphi}\xi^A$). In terms of these quantities we can associate with the pair (Σ_w, P) the null twistor

$$(\Sigma_w, P) \longmapsto W^\Lambda \doteq \left(\xi^A,\, \eta_{A'}\right) \in \mathbb{C}P^3 \tag{3.15}$$

where $\Lambda = 1,2,3,4$ and $\eta_{A'} \doteq -i\,\xi^A y_{AA'}$ is the moment of ξ^A, (with respect to the origin O), evaluated at the marked point P. Conversely, from $\eta_{A'} \doteq -i\,\xi^A y_{AA'}$ we have

$$y_{AA'} = i\left(\xi^B \overline{\eta}_B\right)^{-1} \overline{\eta}_A\, \eta_{A'}, \tag{3.16}$$

and we can parametrize the null line l_w through $y_{AA'}$ according to

$$t \longmapsto X_{AA'}(t) = y_{AA'} + t\, \overline{\xi}_A\, \xi_{A'}. \tag{3.17}$$

Note that, since there is a unique hyperbolic geodesics γ passing through $P \in \Sigma_w$ with endpoint $\gamma(w)$, centre of Σ_w, the twistor W^A can be equivalently thought of as representing γ or, by duality, the Busemann function (3.4) associated with γ. This correspondence between twistors and geodesics in $\mathbb{H}^3$ is particularly useful when dealing with two distinct horospheres Σ_{w_k} and Σ_{w_h}, respectively represented by $\xi^{\,A}(k)\,\overline{\xi}^{\,A'}(k)$ and $\xi^{\,A}(h)\,\overline{\xi}^{\,A'}(h)$. In such a case, to the ordered pair $(\Sigma_{w_k}, \Sigma_{w_h})$ we can associate the null twistor

$$(\Sigma_{w_k}, \Sigma_{w_h}) \longmapsto W^{\,\Lambda}(k,h) \doteq \left(\xi^A(k),\, \overline{\xi}_{\,A'}(h)\right) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \Delta \tag{3.18}$$

where Δ is the diagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$, (which must be removed since the horospheres are distinct). Similarly, we have the correspondence

$$(\Sigma_{w_h}, \Sigma_{w_k}) \longmapsto W^{\,\Lambda}(h,k) \doteq \left(\xi^A(h),\, \overline{\xi}_{\,A'}(k)\right) \tag{3.19}$$

for the pair $(\Sigma_{w_h}, \Sigma_{w_k})$ in reversed order. The characterizations (3.18) and (3.19) are twistorial in the sense that $W^{\Lambda}(h,k)$ and $W^{\Lambda}(k,h)$ are incident, i.e. ,

$$W^{\Lambda}(k,h)\,\overline{W}_{\Lambda}(h,k) = \xi^A(k)\xi_{\,A}(k) + \overline{\xi}_{A'}(h)\overline{\xi}^{\,A'}(h) = 0 \tag{3.20}$$

where $W^{\Lambda}(k,h)\,\overline{W}_{\Lambda}(h,k)$ denotes the (pseudo-hermitian) inner product in twistor space. This implies that we can find two null lines l_k and l_h, with respective tangent vectors $\xi^A(k)\overline{\xi}^{\,A'}(k)$ and $\xi^A(h)\overline{\xi}^{\,A'}(h)$, intersecting each other at a point $x_{AA'}(k,h)$ such that

$$\overline{\xi}_{\,A'}(h) = -i\,\xi^{\,A}(k)x_{AA'}(k,h), \tag{3.21}$$

$$\overline{\xi}_{\,A'}(k) = -i\,\xi^{\,A}(h)x_{AA'}(k,h). \tag{3.22}$$

Formally we can write

$$x_{JJ'}(k,h) = \frac{i}{\xi^A(k)\xi_{\,A}(h)}\left(\xi_{\,J}(k)\overline{\xi}_{J'}(k) + \xi_{\,J}(h)\overline{\xi}_{J'}(h)\right). \tag{3.23}$$

In order to characterize the point $x_{AA'}(k,h)$ in geometrical terms, observe that given any two distinct horospheres Σ_{w_k} and Σ_{w_h} there is a unique parametrized hyperbolic geodesic $t \mapsto \gamma(k,h)(t)$ connecting them. Such a geodesic marks a unique point $y_{(\Sigma_k,h)} \doteq \gamma(k,h) \cap \Sigma_{w_k}$ on Σ_{w_k}, (similarly there is a unique point $y_{(\Sigma_h,k)} \doteq \gamma(h,k) \cap \Sigma_{w_h}$ intercepted on Σ_{w_h} by the orientation-reversed geodesic $\gamma(h,k)$). We let l_k be the null line passing through $y_{(\Sigma_k,h)}$ with tangent vector $\xi^A(k)\overline{\xi}^{\,A'}(k)$, and l_h the null line passing through $y_{(\Sigma_h,k)}$ with tangent $\xi^A(h)\overline{\xi}^{\,A'}(h)$. Both such lines lie in a two-dimensional hyperplane $\subset M^4$ passing through the origin and whose intersection with $\mathbb{H}^3$ traces the geodesic $\gamma(k,h)$. Since the horospheres Σ_{w_k} and Σ_{w_h} are distinct, the lines l_k and l_h necessarily intersect in a point which provides the required $x_{AA'}(k,h)$. Recall that in terms of the spinorial representation $\overrightarrow{w} \longleftrightarrow \xi^{\,A}\,\overline{\xi}^{\,A'}$ and $\overrightarrow{v} \longleftrightarrow \zeta^{\,A}\,\overline{\zeta}^{\,A'}$ of two future-pointing null vector $\overrightarrow{w}$ and $\overrightarrow{v}$ we can write

$$-\langle \overrightarrow{v}, \overrightarrow{w}\rangle = \frac{1}{2}\zeta^{\,A}\,\overline{\zeta}^{\,A'}\xi^{\,B}\,\overline{\xi}^{\,B'}\epsilon_{AB}\,\epsilon_{A'B'} = \frac{1}{2}\zeta_{\,B}\,\overline{\zeta}_{\,B'}\,\xi^{\,B}\,\overline{\xi}^{\,B'}, \tag{3.24}$$

where ϵ_{AB} is the antisymmetric (symplectic) 2-form on spinor space (chosen so that $\epsilon_{01} = 1$ in the selected spin frame), and where spinorial indices are lowered and raised via $\zeta_{\,B} =$

$\zeta^{A}\epsilon_{AB}$ and $\zeta^{A} = \epsilon^{AB}\zeta_{B}$. Applying this to the null vectors $\overrightarrow{w}(k)$ and $\overrightarrow{w}(h)$, defining the two horospheres Σ_{w_k} and Σ_{w_h}, we get the twistorial expression of the corresponding λ-length

$$\lambda(\Sigma_{w_k}, \Sigma_{w_h}) = \sqrt{\frac{1}{2}\,\xi_B(k)\,\overline{\xi}_{B'}(k)\,\xi^{\,B}(h)\,\overline{\xi}^{\,B'}(h)} \doteq \frac{1}{\sqrt{2}}\left|\left|\xi_B(k)\xi^{\,B}(h)\right|\right|. \tag{3.25}$$

In terms of the geodesic $\gamma(k,h)$ we are basically describing the well-known twistor correspondence [29] between the geodesic field of $\mathbb{H}^3$ and the (mini)twistor space $\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \Delta$.

3.2 The computation of lambda-lengths

A key step in discussing the relation among Regge triangulations, twistor theory, and hyperbolic geometry involves the computation of the lambda-lengths (3.7) in terms of the Euclidean lengths of the edges of $\sigma^2(k,h,j)$. To this end, we consider horospheres Σ_k, Σ_h,Σ_j sufficiently near to the vertices $v^0(k)$, $v^0(h)$, $v^0(j)$ of $\sigma^2_{\text{hyp}}(k,h,j)$. We start by evaluating the lambda-lengths (3.7) along the vertical geodesics connecting $v^0(0) \simeq \infty$ with the triangle $\sigma^2_{\text{hyp}}(k,h,j)$. By applying (3.9) and (3.8) we get

$$\begin{aligned}
\lambda\left(\Sigma_\infty, \Sigma_k\right) &= \sqrt{2\frac{t}{z_k}}, \\
\lambda\left(\Sigma_\infty, \Sigma_h\right) &= \sqrt{2\frac{t}{z_h}}, \\
\lambda\left(\Sigma_\infty, \Sigma_j\right) &= \sqrt{2\frac{t}{z_j}},
\end{aligned} \tag{3.26}$$

where $z = z_k$, $z = z_h$, and $z = z_j$ respectively define the z coordinates of the intersection points between the horospheres Σ_k, Σ_h, Σ_j and the corresponding vertical geodesics. Consider now the intersection of the ideal triangle $\sigma^2_{\text{hyp}}(\infty,k,h)$ with each of the horospheres Σ_∞, Σ_k, and Σ_h. Each such an intersection characterizes a corresponding horocyclic segment $\digamma_\infty$, $\digamma_k$, $\digamma_h$ whose hyperbolic length defines (twice) the h-lenght of the horocyclic segment. In particular, the horocyclic segment traced by $\sigma^2_{\text{hyp}}(\infty,k,h) \cap \Sigma_\infty$ is the side $\sigma^1(k,h)$ of the Euclidean triangle $\sigma^2(k,h,j)$. According to (3.9), its h-lenght is provided by

$$K_t\left(k,h\right) = \tanh^{-1}\sqrt{\frac{l^2(k,h)}{l^2(k,h)+4t^2}}. \tag{3.27}$$

On the other hand, the horocyclic segment $\sigma^1(k,h)$ is opposite to the geodesic segment intercepted by the horospheres Σ_k, and Σ_h along the hyperbolic edge $\sigma^1_{\text{hyp}}(k,h)$. The lambda-length of this segment is $\lambda\left(\Sigma_k,\Sigma_h\right)$, and according to a result by R. Penner [12], (Proposition 2.8), among these quantities there holds the relation

$$K_t\left(k,h\right) = \frac{\lambda\left(\Sigma_k,\Sigma_h\right)}{\lambda\left(\Sigma_\infty,\Sigma_k\right)\lambda\left(\Sigma_\infty,\Sigma_h\right)}, \tag{3.28}$$

from which we get

$$\lambda\left(\Sigma_k,\Sigma_h\right) = \frac{2t}{\sqrt{z_k z_h}}\tanh^{-1}\sqrt{\frac{l^2(k,h)}{l^2(k,h)+4t^2}}. \tag{3.29}$$

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Similarly, we compute

$$\lambda\left(\Sigma_h, \Sigma_j\right) = \frac{2t}{\sqrt{z_h z_j}} \tanh^{-1} \sqrt{\frac{l^2(h,j)}{l^2(h,j)+4t^2}}, \tag{3.30}$$

$$\lambda\left(\Sigma_j, \Sigma_k\right) = \frac{2t}{\sqrt{z_j z_k}} \tanh^{-1} \sqrt{\frac{l^2(j,k)}{l^2(j,k)+4t^2}}. \tag{3.31}$$

Note that these relations must hold for any t, and if we take the limit as $t \longrightarrow \infty$ we easily find

$$\lambda\left(\Sigma_k, \Sigma_h\right) = \frac{l(k,h)}{\sqrt{z_k z_h}}, \tag{3.32}$$

$$\lambda\left(\Sigma_h, \Sigma_j\right) = \frac{l(h,j)}{\sqrt{z_h z_j}}, \tag{3.33}$$

$$\lambda\left(\Sigma_j, \Sigma_k\right) = \frac{l(j,k)}{\sqrt{z_j z_k}}. \tag{3.34}$$

We can also compute the h-lengths associated with the decorated ideal triangle $\sigma^2_{\text{hyp}}(k,h,j)$ and defined by

$$H\left(\Sigma_k, \Sigma_h\right) \doteq \frac{\lambda\left(\Sigma_k, \Sigma_h\right)}{\lambda\left(\Sigma_h, \Sigma_j\right)\lambda\left(\Sigma_j, \Sigma_k\right)}, \tag{3.35}$$

$$H\left(\Sigma_h, \Sigma_j\right) \doteq \frac{\lambda\left(\Sigma_h, \Sigma_j\right)}{\lambda\left(\Sigma_j, \Sigma_k\right)\lambda\left(\Sigma_k, \Sigma_h\right)}, \tag{3.36}$$

$$H\left(\Sigma_j, \Sigma_k\right) \doteq \frac{\lambda\left(\Sigma_j, \Sigma_k\right)}{\lambda\left(\Sigma_k, \Sigma_h\right)\lambda\left(\Sigma_h, \Sigma_j\right)}. \tag{3.37}$$

From (3.32) $\div$ (3.34) we get

$$H\left(\Sigma_k, \Sigma_h\right) = \frac{l(k,h)}{l(h,j)l(j,k)}\, z_j, \tag{3.38}$$

$$H\left(\Sigma_h, \Sigma_j\right) = \frac{l(h,j)}{l(j,k)l(k,h)}\, z_k, \tag{3.39}$$

$$H\left(\Sigma_j, \Sigma_k\right) = \frac{l(j,k)}{l(k,h)l(h,j)}\, z_h. \tag{3.40}$$

Since the λ-length $\lambda(\Sigma_k, \Sigma_h)$ can also be expressed in terms of the spinorial quantity $2^{-\frac{1}{2}}||\xi_B(k)\xi^B(h)||$, (see (3.25)), we can use the above connection with the Euclidean lengths $\{l(k,h)\}$ in order to provide, in terms of the spinorial norms $\left\{||\xi_B(k)\xi^B(h)||\right\}$ and of the $\{l(k,h)\}$, the z-coordinates $\{z_k\}$ of the marked points on the horospheres. Explicitly, from the expressions (3.35), (3.38) of the h-length $H(\Sigma_k, \Sigma_h)$ and (3.25) we get

$$H(\Sigma_k, \Sigma_h) = \sqrt{2}\, \frac{||\xi_B(k)\xi^B(h)||}{||\xi_B(h)\xi^B(j)||\ ||\xi_B(j)\xi^B(k)||} = \frac{l(k,h)}{l(h,j)l(j,k)}\, z_j, \tag{3.41}$$

from which we compute

$$z_j = \sqrt{2}\, \frac{||\xi_B(k)\xi^B(h)||}{||\xi_B(h)\xi^B(j)||\ ||\xi_B(j)\xi^B(k)||} \frac{l(h,j)l(j,k)}{l(k,h)}, \tag{3.42}$$

(by a cyclical permutation of (k,h,j) we easily get the expressions for z_h, and z_k). Note that the knowledge of the $\{z_k\}$ provides, up to translations in $\mathbb{R}^2\times\{0\}$, the horospheres $\{\Sigma_k\}$ in $\mathbb{H}^{3,+}_{\text{up}}$ which decorate the vertices of the ideal triangle $\sigma^2_{\text{hyp}}(k,h,j)$, (the actual position of these horospheres is defined by the corresponding null vector $\xi^B(k)\overline{\xi}^{B'}(k)$). It follows from these remarks that from the twistorial decoration of the Euclidean triangle $\sigma^2(k,h,j)$ we can fully recover the horospherically decorated hyperbolic triangle $\sigma^2_{\text{hyp}}(k,h,j)$. In other words we have the correspondance

$$\boxed{\text{Euclidean triangles decorated with null pairwise incident twistors}}$$
$$\Updownarrow$$
$$\boxed{\text{Ideal hyperbolic triangles decorated with horospheres}}$$

This directly bring us to discuss what kind of structure is induced in $\mathbb{H}^{3,+}_{\text{up}}$ by a twistorial field defined over the whole triangulation $|T_l|\to M$.

4. Regge triangulations in twistor space

The geometrical analysis of the previous paragraphs implies that to each of the $N_2(T)$ Euclidean triangles $\sigma^2(k,h,j)$ of $|T_l|\to M$ we can associate a ideal tetrahedron $\sigma^3_{\text{hyp}}(\infty,k,h,j)$ in $\mathbb{H}^{3,+}_{\text{up}}$ and an ideal triangle $\sigma^2_{\text{hyp}}(k,h,j)$ decorated with the horocyclic sectors induced by a choice of horospheres Σ_k, Σ_h, Σ_j. Note that the decoration of the vertices $v^0(k)$, $v^0(h)$, and $v^0(j)$ actually exploits the data of $(z_k,\,\Sigma_k)$, $(z_h,\,\Sigma_h)$, and $(z_j,\,\Sigma_j)$ where the points z_k, z_h, and z_j belongs to the respective horospheres and determine the geodesic γ in $\mathbb{H}^{3,+}_{\text{up}}$ whose endpoint $\gamma(\infty)$ is the centre of Σ_∞. According to (3.15) this decoration of $\sigma^2_{\text{hyp}}(k,h,j)$ can be thought of as induced by the twistorial decoration of the vertices of the Euclidean triangle $\sigma^2(k,h,j)$ defined by the map

$$\begin{aligned}\left\{\sigma^0(i)\right\}_{V(T)} &\longrightarrow \mathbb{C}P^3\\ \sigma^0(k) &\longmapsto \left(\xi^A(k),\eta_{A'}(k)\right),\end{aligned} \tag{4.1}$$

which associates with each vertex $\sigma^0(k)\in|T_l|\to M$ the null twistor describing the marked horosphere (Σ_k,z_k). Equivalently, we can use the decoration defined by the null twistor $W^\Lambda(k,\infty)$ describing the geodesic γ, i.e.,

$$\begin{aligned}\left\{\sigma^0(i)\right\}_{V(T)} &\longrightarrow \mathbb{C}P^1\times\mathbb{C}P^1\backslash\Delta\subset\mathbb{C}P^3\\ \sigma^0(k) &\longmapsto W^\Lambda(k,\infty)\doteq\left(\xi^A(k),\overline{\xi}_{A'}(\infty)\right),\end{aligned} \tag{4.2}$$

where $\xi^A(\infty)\overline{\xi}^{A'}(\infty)$ is the null vector defining the horosphere Σ_∞. Whichever representative we chose, the edges of $\sigma^2(k,h,j)$ carry an induced twistorial decoration defined by

$$\begin{aligned}\left\{\sigma^1(k,h)\right\}_{E(T)} &\longrightarrow \mathbb{C}P^1\times\mathbb{C}P^1\backslash\Delta\subset\mathbb{C}P^3\\ \sigma^1(k,h) &\longmapsto W^\Lambda(k,h)\doteq\left(\xi^A(k),\overline{\xi}_{A'}(h)\right),\end{aligned} \tag{4.3}$$

which to each oriented edge $\sigma^1(k,h)$ of $|T_l| \to M$ associates the null twistor $(\xi^A(k), \overline{\xi}_{A'}(h))$ describing the parametrized geodesic $\gamma(k,h)$. It is worthwhile noticing that the massless twistor fields defined on $|T_l| \to M$ by (4.2) and (4.3) can be equivalently thought of as providing a geometrical realization of an immersion of the random Regge triangulation $|T_l| \to M$ in the quadric $\mathbb{C}P^1 \times \mathbb{C}P^1 \backslash \Delta$ in twistor space $\mathbb{C}P^3$.

As we have seen in paragraph 3.2, the above twistorial decoration allows to associate with each Euclidean triangle $\sigma^2(k,h,j)$ a corresponding ideal triangle $\sigma^2_{\text{hyp}}(k,h,j)$ with a horocyclical decoration of the vertices, which can be recovered in terms of the Euclidean lenghts $\{l(k,h)\}$ of $\sigma^2(k,h,j)$ and of the spinorial norms $\{||\xi_B(k)\xi^B(h)||\}$. Clearly, there is more in such a correspondence, and in particular one is naturally led to explore the possibility of glueing the ideal triangles $\{\sigma^2_{\text{hyp}}(k,h,j)\}$ in the same combinatorial pattern defined by $|T_l| \to M$. This must be done in such a way that the twistor fields on $|T_l| \to M$ provide a consistent horocyclical decoration of the vertices of the ideal triangulation defined by $\{\sigma^2_{\text{hyp}}(k,h,j)\}$. In performing such an operation one must take care of three basic facts: *(i)* ideal triangles are rigid since any two of them are congruent; *(ii)* the adjacent sides of two ideal triangles can be identified up to the freedom of performing an arbitrary traslation along the edges, (each edge of an ideal triangle is isometric to the real line, its hyperbolic lenght being infinite, and two adjacent edges may freely slide one past another); *(iii)* Since $\mathbb{H}^3$ is a space of left cosets of SU(2) in $Sl(2,\mathbb{C})$, the identification of the marked point z_k on the horosphere (Σ_k, z_k), associated with a vertex $\sigma^0(k)$, is only defined up to the action of SU(2). These translational and SU(2) degrees of freedom can be exploited in order to specifing how the decoration provided by the horocyclic sectors in an ideal triangle is extended to the adjacent ideal triangle. Within such a set-up, let us consider the star $Star[\sigma^0(k)]$ of a generic vertex $\sigma^0(k)$ over which $q(k)$ triangles $\sigma^2(k,h_\alpha,h_{\alpha+1})$ are incident. Let us label the corresponding set of hyperbolic ideal triangles by $\sigma^2_{\text{hyp}}(k,h_\alpha,h_{\alpha+1})$, $\alpha = 1,\ldots,q(k)$, with $h_\alpha = h_\beta$ if $\beta = \alpha \mod q(k)$. The natural hyperbolic structure on

$$P^2(k) \doteq \bigcup_{\alpha=1}^{q(k)} \sigma^2_{\text{hyp}}(k,h_\alpha,h_{\alpha+1}) - \{v^0(k)\}\,, \tag{4.4}$$

($v^0(k)$ being the vertex associated with $\sigma^0(k)$), induces a similarity structure on the link associated with $v^0(k)$

$$link\ \left[v^0(k)\right] \doteq \bigcup_{\alpha=1}^{q(k)} \sigma^1_{\text{hyp}}(h_\alpha,h_{\alpha+1}), \tag{4.5}$$

which characterizes, as k varies, the hyperbolic surface one gets by glueing the hyperbolic triangles $\sigma^2_{\text{hyp}}(k,h_\alpha,h_{\alpha+1})$. To determine such a similarity structure, let us consider a triangle, say $\sigma^2_{\text{hyp}}(k,h_1,h_2)$, in $P^2(k)$, and let $\digamma^{h_1}_k$ be the oriented horocyclic segment cut in $\sigma^2_{\text{hyp}}(k,h_1,h_2)$ by the horosphere Σ_k. This horocyclic segment can be extended, in a counterclockwise order, to the other $q(k)-1$ ideal triangles in the set $\{\sigma^2_{\text{hyp}}(k,h_\alpha,h_{\alpha+1})\}$ by requiring that such an extension meets orthogonally each adjacent geodesic side of the $q(k)$ triangles considered. Since the horospheres are congruent and the identification between adjacent sides of ideal triangles is only defined up to a shift, such an extension procedure generates a sequence of $q(k)$ horocyclic segments $\{\digamma_k^{h_\alpha}\}$ which eventually re-enters the triangle $\sigma^2_{\text{hyp}}(k,h_1,h_2)$ with a horocyclic segment $\widehat{\digamma}_k^{h_{q(k)}}$ which will be parallel to $\digamma_k^{h_1}$ but not necessarily coincident with it. The similarity structure is completely characterized by the Euclidean similarity $f:\mathbb{R}\to\mathbb{R}$ which maps, along $\sigma^1_{\text{hyp}}(h_1,h_2)$, the point $\digamma_k^{h_1}\cap\sigma^1_{\text{hyp}}(h_1,h_2)$ to the point $\widehat{\digamma}_k^{h_{q(k)}}\cap\sigma^1_{\text{hyp}}(h_1,h_2)$. The horocycle curve $t\mapsto\digamma_k(t)$, $0\le t\le 2\pi$ closes up, i.e. , $\digamma_k^{h_1}\cap\sigma^1_{\text{hyp}}(h_1,h_2)=\widehat{\digamma}_k^{h_{q(k)}}\cap\sigma^1_{\text{hyp}}(h_1,h_2)$ iff the Euclidean length $|\digamma_k(t)|_{\text{Euc}}$ of $t\mapsto\digamma_k(t)$ is 2π, (note that $|\digamma_k(t)|_{\text{Euc}}$ is always a constant). In our case, the Euclidean length of the horocycle curve $t\mapsto\digamma_k(t)$, $0\le t\le 2\pi$ is given by the conical defect $\Theta(k)=\Sigma_{\alpha=1}^{q(k)}\theta_{\alpha+1,k,\alpha}$ supported at the vertex $\sigma^0(k)\in|T_l|\to M$. Thus, the similarity ratio is given by $(\frac{\Theta(k)}{2\pi})$. Given such a ratio one can compute, by exploiting (3.10), the signed hyperbolic distance between the horocycle segments $\digamma_k^{h_1}$ and $\widehat{\digamma}_k^{h_{q(k)}}$ according to

$$\mp d_{\mathbb{H}^3}(\digamma_k^{h_1},\,\widehat{\digamma}_k^{h_{q(k)}})=\ln\frac{\Theta(k)}{2\pi}\doteq d[v^0(k)],\tag{4.6}$$

where the sign is chosen to be positive iff $\Theta(k)<2\pi$, i.e. if the horodisk sector bounded by $\digamma_k^{h_1}$ contains the sector bounded by $\widehat{\digamma}_k^{h_{q(k)}}$. Note that the number $d[v^0(k)]$ does not depend from the initial choice of $\digamma_{v^0(k)}^{h_1}$, and is an invariant only related to the conical defect $\Theta(k)$ supported at the vertex $\sigma^0(k)$ of $|T_l|\to M$. It can be identified with the invariant introduced by W. Thurston [13] in order to characterize the completeness of the hyperbolic structure of a surface obtained by gluing hyperbolic ideal triangles (the structure being complete iff the invariants $d[v^0(k)]$ are all zero for each ideal vertex $v^0(k)$). A classical result by W. Thurston, ([13], prop. 3.10.2), implies that the glueing of the $N_2(T)$ ideal triangles according to the procedure just described gives rise to an open hyperbolic surface Ω with geodesic boundaries. Each boundary component $\partial\Omega_k$ is associated with a corresponding vertex $\sigma^0(k)$ of $|T_l|\to M$, and has a length provided by

$$|\partial\Omega_k|=\left|d(v^0(k))\right|=\left|\ln\frac{\Theta(k)}{2\pi}\right|.\tag{4.7}$$

These geodesic boundaries come also endowed with a SU(2) holonomy which is generated by the twistors $\left(\xi^A(k),\eta_{A'}(k)\right)$ decorating each vertex $\sigma^0(k)$. Explicitly, let

$$\left(\xi^A(k),\,\eta_{A'}(k)\right)_{\sigma^2(k,h_\alpha,h_{\alpha+1})}\tag{4.8}$$

denote the twistor associated with the marked horosphere (Σ_k,z_k) which decorates the vertex $v^0(k)$ of the triangle $\sigma^2_{\text{hyp}}(k,h_\alpha,h_{\alpha+1})$. If we denote by $I\doteq\frac{1}{\sqrt{2}}\delta_{AA'}$ the hermitian matrix corresponding to $(1,0,0,0)\in\mathbb{H}^3$, then we can set $\eta_{A'}(k)\doteq -i\,\xi^{\,A}(k)\,z_{AA'}(k)$ where $z_{AA'}(k)$ is the $\mathrm{SL}(2,\mathbb{C})$ matrix associated with the marked point z_k. When we move from the triangle $\sigma^2(k,h_\alpha,h_{\alpha+1})$ to the adjacent one $\sigma^2(k,h_{\alpha+1},h_{\alpha+2})$ the corresponding group elements $z_{AA'}(k)|_{\sigma^2(k,h_\alpha,h_{\alpha+1})}$ and $z_{AA'}(k)|_{\sigma^2(k,h_{\alpha+1},h_{\alpha+2})}$, being associated with the same coset $\{z_k\}\in\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$, are related by

$$z_{AA'}(k)|_{\sigma^2(k,h_{\alpha+1},h_{\alpha+2})}\;=z_{AA'}(k)|_{\sigma^2(k,h_\alpha,h_{\alpha+1})}\,s(k,h_{\alpha+1}),\tag{4.9}$$

where $s(k, h_{\alpha+1}) \in \mathrm{SU}(2)$, and where the labelling $(k, h_{\alpha+1})$ refers to the edge $\sigma^1(k, h_{\alpha+1})$ shared between the two adjacent triangle $\sigma^2(k, h_\alpha, h_{\alpha+1})$ and $\sigma^2(k, h_{\alpha+1}, h_{\alpha+2})$. Since the locally Euclidean structure in each $Star[\sigma^0(k)]$ is characterized by a $\mathrm{U}(1)$ holonomy $e^{i\,\Theta(k)}$, we require that $s(k, h_{\alpha+1})$ lies in the maximal torus in SU(2), i.e.

$$s(k, h_{\alpha+1}) = e^{i\sigma_3\,\theta_{\alpha+1,k,\alpha}} \doteq \begin{pmatrix} e^{i\,\theta_{\alpha+1,k,\alpha}} & 0 \\ 0 & e^{-i\,\theta_{\alpha+1,k,\alpha}} \end{pmatrix}. \tag{4.10}$$

Thus, by circling around the star $Star[\sigma^0(k)]$ we get

$$z_{AA'}(k)|_{\sigma^2(k,h_{q(k)},h_1)} = z_{AA'}(k)|_{\sigma^2(k,h_1,h_2)} \prod_{\alpha=1}^{q(k)} s(k, h_{\alpha+1}), \tag{4.11}$$

with $h_{q(k)+1} = h_1$. The group element defined by

$$U_k \doteq \prod_{\alpha=1}^{q(k)} s(k, h_{\alpha+1}) = e^{i\sigma_3\,\Theta(k)} \tag{4.12}$$

provides the SU(2) holonomy associated with the geodesic boundary $\partial\Omega_k$. Associated with such a holonomy we have $\mathfrak{su}(2)$-valued flat gauge potentials $A_{(k)}$ locally defined by

$$A_{(k)} \doteq \frac{i}{4\pi}|\partial\Omega_k|\gamma_k\,(\Theta(k)\ \sigma_3)\,\gamma_k^{-1}\left(\frac{d\zeta(k)}{\zeta(k)} - \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)}\right), \tag{4.13}$$

where $\zeta(k)$ is a complex coordinate in a neighborhood of the boundary component $\partial\Omega_k$, (defined by glueing to $\partial\Omega_k$ a punctured disk $\in \mathbb{C}$ with coordinate $\zeta(k)$), and where $\gamma_i \in \mathrm{SU}(2)$. We can sum up these remarks in the following

Proposition 1. *A closed random Regge surface* $(|T_l| \to M)$ *whose vertex set* $\{\sigma^0(k)\}_1^{V(T)}$ *is decorated with the null twistor field* $\sigma^0(k) \longmapsto W^\Lambda(k,\infty)$ *has a dual description as the ideal triangulation* $H\,(|T_l| \to M)$ *of an open hyperbolic surface* $\Omega \sim M/V(T)$ *with geodesic boundaries* $\partial\Omega_k$ *of length* $|\partial\Omega_k| = |\ln\frac{\Theta(k)}{2\pi}|$. *To any such a boundary it is associated a* SU(2) *holonomy* $U_k = e^{i\sigma_3\Theta(k)}$ *generated by a* $\mathfrak{su}(2)$*-valued flat gauge potential* $A_{(k)}$.

4.1 Moduli kinematics of closed/open duality

Let $\mathcal{T}_{g,N_0}(L)$ denote the Teichmüller space of hyperbolic surfaces Ω with geodesic boundary components of length

$$L = (L_1, \ldots, L_{N_0}) \doteq (|\partial\Omega_1|, \ldots, |\partial\Omega_{N_0}|) \in \mathbb{R}_+^{N_0}. \tag{4.14}$$

Note that, by convention, a boundary component such that $|\partial\Omega_j| = 0$ is a cusp and moreover $\mathcal{T}_{g,N_0}(L = 0) = \mathcal{T}_{g,N_0}$, where $\mathcal{T}_{g,N_0}$ is the Teichmüller space of hyperbolic surfaces with N_0 punctures, (with $6g - 6 + 2N_0 \geq 0$). The elements of $\mathcal{T}_{g,N_0}(L)$ are marked Riemann surface modelled on a surface S_{g,N_0} of genus g with complete finite-area metric of constant Gauss curvature -1, (and with N_0 geodesic boundary components $\partial S = \sqcup\partial S_j$

of fixed length), i.e. , a triple $(S_{g,N_0},\, f,\, \Omega)$ where $f\,:\, S_{g,N_0}\ \to \Omega$ is a quasiconformal homeomorphism, (the marking map), which extends uniquely to a homeomorphism from $S_{g,N_0}\cup\partial S$ onto $\Omega\cup\partial\Omega$. Any two such a triple $(S_{g,N_0},\, f_1,\, \Omega_{(1)})$ and $(S_{g,N_0},\, f_2,\, \Omega_{(2)})$ are considered equivalent iff there is a biholomorphism $h:\Omega_{(1)}\to\Omega_{(2)}$ such that $f_2^{-1}\circ h\circ f_1$: $S_{g,N_0}\cup\partial S\to S_{g,N_0}\cup\partial S$ is homotopic to the identity via continuous mappings pointwise fixing ∂S. For each given string $L=(L_1,\ldots,L_{N_0})$ there is a natural action on $\mathcal{T}_{g,N_0}(L)$ of the mapping class group $\mathcal{M}ap_{g,N_0}$ defined by the group of all the isotopy classes of orientation preserving homeomorphisms of Ω which leave each boundary component $\partial\Omega_j$ pointwise (and isotopy-wise) fixed. This action changes the marking f of S_{g,N_0} on Ω, and characterizes the quotient space

$$\mathcal{M}_{g,N_0}(L)\doteq\frac{\mathcal{T}_{g,N_0}(L)}{\mathcal{M}ap_{g,N_0}} \tag{4.15}$$

as the moduli space of Riemann surfaces (homeomorphic to S_{g,N_0}) with N_0 boundary components of length $|\partial\Omega_j|=L_j$. Note again that when $\{L_j\to 0\}$, $\mathcal{M}_{g,N_0}(L)$ reduces to the usual moduli space $\mathcal{M}_{g,N_0}$ of Riemann surfaces of genus g with N_0 punctures. We have $dim_{\mathbb{R}}\mathcal{M}_{g,N_0}(L)=6g-6+3N_0$ and $dim_{\mathbb{R}}\mathcal{M}_{g,N_0}=6g-6+2N_0$ the extra N_0 coming from the boundary lengths. Let us denote by $\overline{\mathcal{M}}_{g,N_0}$ the Deligne-Mumford compactification of the moduli space of N_0-pointed closed surfaces of genus g. As a connected complex obifold $\overline{\mathcal{M}}_{g,N_0}$ is naturally endowed with the i-th tautological line bundle $\mathcal{L}(i)$ whose fiber at the point $(\Omega,p_1,\ldots,p_{N_0})\in\overline{\mathcal{M}}_{g,N_0}$ is the cotangent space of Ω at p_i. Let us recall also that for surfaces with punctures $\in\mathcal{T}_{g,N_0}$ one can introduce a trivial bundle, Penner's decorated Teichmüller space,

$$\widetilde{\mathcal{T}}_{g,N_0}\xrightarrow{\pi_{\rm hor}}\mathcal{T}_{g,N_0} \tag{4.16}$$

whose fiber over a punctured surface $\widetilde{\Omega}$ is the set of all N_0-tuples of horocycles in $\widetilde{\Omega}$, with one horocycle around each puncture, (there is a corresponding trivial fibration $\widetilde{\mathcal{M}}_{g,N_0}$ over $\mathcal{M}_{g,N_0}$). A section of this fibration is defined by choosing the total length of the horocycle assigned to each puncture in $\widetilde{\Omega}$.

In our case, the hyperbolic surfaces Ω with geodesic boundary $\in\mathcal{T}_{g,N_0}(L)$ arise from the interplay between the geometry of the horocycles and the conical holonomies $e^{i\sigma_3\Theta(k)}$ of the underlying Regge triangulation $|T_l|\to M$. In particular, if let $\{\Theta(k)\to 2\pi\}$, then there is a natural mapping between $(\mathbb{L}^+)^{N_0}\times\mathcal{T}_{g,N_0}(L)$, ($\mathbb{L}^+$ being the future light cone parametrizing the horospheres), and the decorated Teichmüller space $\widetilde{\mathcal{T}}_{g,N_0}$. This *conical forgetful* mapping is defined by associating to Ω the hyperbolic surface $\widetilde{\Omega}$ with N_0 punctures obtained by letting $\{\Theta(k)\to 2\pi\}$ and by decorating the resulting cusps with the horocycles traced by the horospheres $\{\Sigma_k\}$, i.e.

$$\begin{gathered}(\mathbb{L}^+)^{N_0}\times\mathcal{T}_{g,N_0}(L)\longrightarrow\widetilde{\mathcal{T}}_{g,N_0}\\ \left(\{\Sigma_k\},\Omega\cup\left\{\sqcup_{k=1}^{N_0(T)}(\partial\Omega_k,|\ln\frac{\Theta(k)}{2\pi}|\right\}\right)\longmapsto\widetilde{\Omega}\doteq\left(\{\Sigma_k\},\Omega\simeq M\setminus V(T)\right),\end{gathered} \tag{4.17}$$

where $V(T)$ denotes the set of $N_0(T)$ vertex of M. In such a construction an interesting role is played by the lambda-lengths associated with the decorated edges of the triangles

$\sigma^2_{\rm hyp}$. According to a classical result by Penner ([30], Theorem 3.3.6), the pull-back $\pi^*_{\rm hor}\omega_{\rm WP}$ under the map $\pi_{\rm hor}: \widetilde{\mathcal{T}}^{\,s}_g \to \mathcal{T}^s_g$ of the Weil-Petersson Kähler two-form $\omega_{\rm WP}$ is given by

$$-2\sum_{[\sigma^2_{\rm hyp}]} d\ln\lambda_0\wedge d\ln\lambda_1 + d\ln\lambda_1\wedge d\ln\lambda_2 + d\ln\lambda_2\wedge d\ln\lambda_0\,, \tag{4.18}$$

where the sum runs over all ideal triangles $\sigma^2_{\rm hyp}$ whose ordered edges take the lambda-lengths λ_0, λ_1, λ_2. Note that (for dimensional reason) $\pi^*\omega_{\rm WP}$ is a degenerate pre-symplectic form. Either by pulling back $\pi^*_{\rm hor}\omega_{\rm WP}$ one more time under the action of the conical forgetful mapping (4.17), or by analyzing its invariance properties under the mapping class group, is straightforward to verify that (4.18) extends also to bordered case. In our setting it provides

$$\begin{aligned}\pi^*_{\rm hor}\omega_{\rm WP}(\Sigma) = -2 &\sum_{[\sigma^2_{\rm hyp}]_{F(T)}} d\ln\lambda\left(\Sigma_k,\Sigma_h\right)\wedge d\ln\lambda\left(\Sigma_h,\Sigma_j\right) + \\ &+ d\ln\lambda\left(\Sigma_h,\Sigma_j\right)\wedge d\ln\lambda\left(\Sigma_j,\Sigma_k\right) + d\ln\lambda\left(\Sigma_j,\Sigma_k\right)\wedge d\ln\lambda\left(\Sigma_k,\Sigma_h\right) \\ = -2&\sum_{F(t)} \frac{dl(k,h)\wedge dl(h,j)}{l(k,h)l(h,j)} + \frac{dl(h,j)\wedge dl(j,k)}{l(h,j)l(j,k)} + \frac{dl(j,k)\wedge dl(k,h)}{l(j,k)l(k,h)}\,,\end{aligned} \tag{4.19}$$

where we have exploited the expressions (3.32), (3.33), (3.34) providing the lambda-lengths in terms of the Euclidean edge-lengths $l(k,h)$ of the Regge triangulation.

If we denote by $V_{g,N_0}\left(\mathcal{M}_{g,N_0}(L)\right)$ the volume of the moduli space $\mathcal{M}_{g,N_0}(L)$ with respect to the measure associated with the Weil-Petersson form $\omega_{\rm WP}(\Sigma)$, then one can compute the dependence of $V_{g,N_0}\left(\mathcal{M}_{g,N_0}(L)\right)$ from the boundary lengths $L=(L_1,\ldots,L_{N_0})$ by exploiting a remarkable result due to M. Mirzakhani [15, 16]

Theorem 2. *(Maryam Mirzakhani (2003))*
The Weil-Petersson volume $V_{g,N_0}\left(\mathcal{M}_{g,N_0}(L)\right)$ is a polynomial in $L_1,\ldots,L_{N_0}$

$$V_{g,N_0}\left(\mathcal{M}_{g,N_0}(L)\right) = \sum_{\substack{(\alpha_1,\ldots,\alpha_{N_0})\in(\mathbb{Z}_{\geq 0})^{N_0}\\ |\alpha|\leq 3g-3+N_0}} C_{\alpha_1\ldots\alpha_{N_0}} L_1^{2\alpha_1}\ldots L_{N_0}^{2\alpha_{N_0}}\,, \tag{4.20}$$

where $|\alpha| = \sum_{i=1}^{N_0}\alpha_i$ and where the coefficients $C_{\alpha_1\ldots\alpha_{N_0}} > 0$ are (recursively determined) numbers of the form

$$C_{\alpha_1\ldots\alpha_{N_0}} = \pi^{6g-6+2N_0-2|\alpha|}\cdot q \tag{4.21}$$

for rationals $q\in\mathbb{Q}$.

Moreover Mirzakhani [15, 16] is also able to express $C_{\alpha_1\ldots\alpha_{N_0}}$ in terms of the intersection numbers $<\tau_{\alpha_1}\ldots\tau_{\alpha_{N_0}}>$ [31] of the tautological line bundles $\mathcal{L}(i)$ over $\overline{\mathcal{M}}_{g,N_0}$ according to

$$C_{\alpha_1\ldots\alpha_{N_0}} = \frac{2^{m(g,N_0)|\alpha|}}{2^{|\alpha|}\prod_{i=1}^{N_0}\alpha_i!(3g-3+N_0-|\alpha|)!} <\tau_{\alpha_1}\ldots\tau_{\alpha_{N_0}}>, \tag{4.22}$$

$$<\tau_{\alpha_1}\ldots\tau_{\alpha_{N_0}}> \doteq \int_{\overline{\mathcal{M}}_{g,N_0}} \psi_1^{\alpha_1}\ldots\psi_{N_0}^{\alpha_{N_0}}\cdot\omega_{\rm WP}^{3g-3+N_0-|\alpha|} \tag{4.23}$$

where ψ_i is the first Chern class of $\mathcal{L}(i)$, and where $m(g, N_0) \doteq \delta_{g,1}\delta_{N_0,1}$. Note in particular that the constant term $C_{0\ldots 0}$ of the polynomial $V_{g,N_0}(\mathcal{M}_{g,N_0}(L))$ is the volume of $\overline{\mathcal{M}}_{g,N_0}$ i.e. ,

$$C_{0\ldots 0} = V_{g,N_0}\left(\overline{\mathcal{M}}_{g,N_0}\right) = \int_{\overline{\mathfrak{M}}_{g,N_0}} \frac{\omega_{\mathrm{WP}}{}^{3g-3+N_0(T)}}{(3g-3+N_0(T))!} \tag{4.24}$$

The valutation of the $C_{\alpha_1\ldots\alpha_{N_0}}$ is at fixed N_0 (and at fixed genus g), and it is interesting to compare it to what is known when the genus increases or when N_0 increases. In particular, let us recall that the Weil-Petersson volume of the moduli space $\overline{\mathcal{M}}_{g,N_0}$ for any fixed value of N_0 is such that

$$A_1^g(2g)! \leq V_{g,N_0}\left(\overline{\mathcal{M}}_{g,N_0}\right) \leq A_2^g(2g)!, \tag{4.25}$$

where the constants $0 < A_1 < A_2$ are independent of N_0 (see [32, 33]). Conversely, the large N_0 asymptotics of $V_{g,N_0}\left(\overline{\mathcal{M}}_{g,N_0}\right)$ at fixed genus has been discussed by Manin and Zograf [34, 35]. They obtained the asymptotic series

$$V_{g,N_0}\left(\overline{\mathcal{M}}_{g,N_0}\right) = \pi^{6g-6+2N_0} \times (N_0+1)^{\frac{5g-7}{2}} C^{-N_0} \left(B_g + \sum_{k=1}^{\infty} \frac{B_{g,k}}{(N_0+1)^k}\right), \tag{4.26}$$

where $C = -\frac{1}{2} j_0 \frac{d}{dz} J_0(z)|_{z=j_0}$, ($J_0(z)$ the Bessel function, j_0 its first positive zero); (note that $C \simeq 0.625\ldots$.). The genus dependent parameters B_g are explicitly given [35] by

$$\begin{cases} B_0 = \frac{1}{A^{1/2}\Gamma(-\frac{1}{2})C^{1/2}}, & B_1 = \frac{1}{48}, \\ B_g = \frac{A^{\frac{g-1}{2}}}{2^{2g-2}(3g-3)!\Gamma(\frac{5g-5}{2})C^{\frac{5g-5}{2}}} \left\langle \tau_2^{3g-3} \right\rangle, & g \geq 2 \end{cases} \tag{4.27}$$

where $A \doteq -j_0^{-1} J_0'(j_0)$, and $\left\langle \tau_2^{3g-3} \right\rangle$ is a Kontsevich-Witten [36] intersection number, (the coefficients $B_{g,k}$ can be computed similarly-see [35] for details).

4.2 An example of open/closed string duality

The preceding results provide a suitable kinematical set up for establishing a open/closed string duality once the appropriate field decoration is activated. To this end let us consider the non-dynamical null twistors fields decorating the vertex of the Regge triangulation. These fields geometrically describe geodesics in $\mathbb{H}^3$, with an end point at $\infty \in \partial\mathbb{H}^{3,+}_{\mathrm{up}} = (\mathbb{R}^2 \times \{0\}) \cup \{\infty\}$, projecting to the N components $\partial\Omega_k$ of the boundary of Ω. Thus, they can be interpreted as fields on Ω with preassigned Dirichlet conditions on the various boundary components $\partial\Omega_k$, and we can consider the N-point function on $\mathcal{M}_{g,N_0}(L)$, describing correlations between such Dirichlet conditions. Explicitly, let us consider the λ-lengths $\lambda(\Sigma_\infty, \Sigma_k)$ associated with the vertical geodesic connecting $v^0(0) \simeq \infty$ with the generic vertex $v^0(k)$ of the ideal triangulation $H(|T| \to M)$. We form the expression

$$Z^{\text{open}}_{N_0,g}\left((L_1, \delta(\Sigma_\infty, \Sigma_1); \ldots; (L_{N_0}, \delta(\Sigma_\infty, \Sigma_{N_0})\right) \doteq \tag{4.28}$$
$$= \sum_{\substack{(\alpha_1,\ldots,\alpha_{N_0}) \in (\mathbb{Z}_{\geq 0})^{N_0} \\ |\alpha| \leq 3g-3+N_0}} C_{\alpha_1\ldots\alpha_{N_0}} \left[\lambda(\Sigma_\infty, \Sigma_1)\, L_1\right]^{2\alpha_1} \ldots \left[\lambda(\Sigma_\infty, \Sigma_{N_0})\, L_{N_0}\right]^{2\alpha_{N_0}}$$
$$= 2^{|\alpha|} \sum_{\substack{(\alpha_1,\ldots,\alpha_{N_0}) \in (\mathbb{Z}_{\geq 0})^{N_0} \\ |\alpha| \leq 3g-3+N_0}} C_{\alpha_1\ldots\alpha_{N_0}}\, e^{\alpha_1\, \delta(\Sigma_\infty, \Sigma_1)}\, L_1^{2\alpha_1} \ldots e^{\alpha_{N_0}\, \delta(\Sigma_\infty, \Sigma_{N_0})}\, L_{N_0}^{2\alpha_{N_0}},$$

where $\delta(\Sigma_\infty, \Sigma_k)$ is the signed hyperbolic distance between the respective horosphere. Thus, $Z^{\text{open}}_{N_0,g}\left(; \delta(\Sigma_\infty, \Sigma_1), \ldots, \right)$ basically provides correlations in the moduli space $\overline{\mathcal{M}}_{g,N_0}(L)$ among the Dirichlet boundary conditions, along the $\{\partial\Omega_k\}$ boundary components, of the local fields $\delta(\Sigma_\infty, \Sigma_k)$. Such correlations describe the distribution in $\overline{\mathcal{M}}_{g,N_0}(L)$ of the (hyperbolic) distance from the surfaces $\Omega \in \overline{\mathcal{M}}_{g,N_0}(L)$ and the Euclidean screen $\Sigma_\infty \subset \mathbb{H}^3_{\text{hyp}}$ from which the generic hyperbolic surface Ω is locally generated by projecting Euclidean triangles into hyperbolic triangles. From Mirzakhani's results we get

$$Z^{\text{open}}_{N_0,g}\left((L_1, \delta(\Sigma_\infty, \Sigma_1); \ldots\right) = \frac{1}{(3g-3+N_0-|\alpha|)!} \times \tag{4.29}$$
$$\times \sum_{\substack{(\alpha_1,\ldots,\alpha_{N_0}) \in (\mathbb{Z}_{\geq 0})^{N_0} \\ |\alpha| \leq 3g-3+N_0}} \int_{\overline{\mathcal{M}}_{g,N_0}} \prod_{i=1}^{N_0} \frac{L_i^{2\alpha_i}\, e^{\alpha_i\, \delta(\Sigma_\infty, \Sigma_i)}}{\alpha_i!} \psi_i^{\alpha_i}\, \omega_{WP}^{3g-3+N_0-|\alpha|}.$$

We could insert here the explicit expression of the boundary lengths $L_i = \left(\ln \frac{\Theta(i)}{2\pi}\right)$, however, for our purposes it is more interesting to consider the scaling regime in which the projection field $\{\delta(\Sigma_\infty, \Sigma_k)\}$ generates the bordered hyperbolic surface Ω from the N_0-pointed closed surface $M \backslash V(T)$ associated with the random Regge triangulation $|T_l| \to M$. According to formula (31) governing the distance scaling in hyperbolic three-geometry, such a regime corresponds all possible rescalings $\delta(\Sigma_\infty, \Sigma_k) \to \beta\, \delta(\Sigma_\infty, \Sigma_k)$ and $L_k \to \beta^{-1} L_k$, $\beta \in (0, \infty)$, of the hyperbolic distance and the boundary lengths, such that

$$L_k(\beta) e^{\delta_\beta(\Sigma_\infty, \Sigma_k)} = \left|\ln \frac{\Theta(k)}{2\pi}\right| \doteq t_k^{\frac{1}{2}}, \quad \beta \in (0, \infty). \tag{4.30}$$

Under such regime, we get

$$Z^{\text{open}}_{N_0,g}\left((L_1(\beta), \delta_\beta(\Sigma_\infty, \Sigma_1); \ldots\right) = Z^{\text{closed}}_{N_0,g}\left(t_1 \ldots t_{N_0}\right) \tag{4.31}$$
$$\doteq \frac{1}{(3g-3+N_0-|\alpha|)!} \sum_{\substack{(\alpha_1,\ldots,\alpha_{N_0}) \in (\mathbb{Z}_{\geq 0})^{N_0} \\ |\alpha| \leq 3g-3+N_0}} \int_{\overline{\mathcal{M}}_{g,N_0}} \prod_{i=1}^{N_0} \frac{t_i^{\alpha_i}}{\alpha_i!} \psi_i^{\alpha_i}\, \omega_{WP}^{3g-3+N_0-|\alpha|},$$

which is the generating function, at finite N_0 and at finite genus g, of the intersection theory [31, 37] over the moduli space $\overline{\mathcal{M}}_{g,N_0}$ of closed N_0-pointed genus g surfaces $M \backslash V(T)$ associated with random Regge triangulations $|T_l| \to M$. Note that the Weil-Peterson form ω_{WP} in (4.31) can be appropriately interpreted in the simplicial form (4.19). The open/closed surface duality mapping (4.31) extends to the moduli spaces $\overline{\mathcal{M}}_{g,N_0}(L)$ and $\overline{\mathcal{M}}_{g,N_0}$ the geometric duality between the hyperbolic (ideally triangulated surface) with geodesic boundary $\Omega = H(|T| \rightarrow M)$ and the closed N_0-pointed surface $\widetilde{\Omega}$ associated with a random Regge triangulation. Note that the duality (4.31) can be immediately rephrased in twistorial terms by recalling the connection (3.25) between the λ-lengths and the associated null twistors in $\mathbb{C}P^1 \times \mathbb{C}P^1 \backslash \Delta$ given by

$$\lambda(\Sigma_\infty, \Sigma_k) = \frac{1}{\sqrt{2}} \left\| \xi_B(\infty)\xi^B(k) \right\|, \tag{4.32}$$

where $W^\Lambda(\infty, k) \doteq (\xi^A(\infty), \overline{\xi}_{A'}(k))$ is the null twistor corresponding to the unique geodetic in $\mathbb{H}^3$ connecting the two horospheres Σ_∞ and Σ_k. We get

$$Z^{open}_{N_0,g}((L_1, \delta(\Sigma_\infty, \Sigma_1)); \ldots) = Z^{open}_{N_0,g}((L_1, W^\Lambda(\infty, k)); \ldots), \tag{4.33}$$

where

$$Z^{open}_{N_0,g}((L_1, W^\Lambda(\infty, k)); \ldots) = \frac{1}{2^{2|\alpha|}(3g-3+N_0-|\alpha|)!} \times \tag{4.34}$$

$$\times \sum_{\substack{(\alpha_i)\in(\mathbb{Z}_{\geq 0})^{N_0} \\ |\alpha|\leq 3g-3+N_0}} \int_{\overline{\mathcal{M}}_{g,N_0}} \prod_{i=1}^{N_0} \frac{L_i^{2\alpha_i} \left\| \xi_B(\infty)\xi^B(i) \right\|^{2\alpha_i}}{\alpha_i!} \psi_i^{\alpha_i} \, \omega_{WP}^{3g-3+N_0-|\alpha|},$$

and the open/closed duality can be interpreted in terms of the scaling behavior of $L_i^{2\alpha_i} \left\| \xi_B(\infty)\xi^B(i) \right\|^{2\alpha_i}$, i.e. , of twistorial field insertions at the vertices of the closed pointed surface associated with $|T_l| \rightarrow M$.

In order to describe a full-fledged open/closed string duality in such a setting let us recall that, according to the discussion in the preceding paragraph, the hyperbolic bordered surface Ω carries the gauge degrees of freedom associated with the flat SU(2) connection $A_{(k)}$ defined by (4.13). We can promote such $A_{(k)}$ to be dynamical fields and consider the amplitude of the corresponding SU(2) Yang-Mills theory. For a surface Ω of genus g, area $A(\Omega)$, and $N_0(T)$ geodesic boundary components $\{\partial\Omega_k\}$ with given holonomies $\{U_k = e^{i\sigma_3\Theta(k)}\}$, such an amplitude is given by the celebrated formula (see e.g., [38])

$$Z^{Y-M}(U_1, \ldots, U_{\mathrm{No}}; A(\Omega)) = \sum_{R\in\{Unitary\ Irreps\}} (\dim R)^{2-2g-N_0} \, e^{-\beta^2_{\mathrm{YM}} A(\Omega)\, C_2(R)} \prod_{i=1}^{N_0} \chi_R(U_i), \tag{4.35}$$

where the sum runs over all unitary irreducible representations of SU(2), $\chi_R(U_i)$ are the associated characters, $C_2(R)$ is the eigenvalue of the quadratic Casimir in the representation R, and finally, β^2_{YM} is the gauge coupling. Since we are considering a hyperbolic surface with geodesic boundary, by the Gauss-Bonnet theorem

$$\int_\Omega K_\Omega \, dA + \int_{\partial\Omega} k_{\partial\Omega} \, ds = 2\pi\,(2 - 2g - N_0), \tag{4.36}$$

(where $K_\Omega = -1$ and $k_{\partial\Omega} = 0$ are the Gaussian curvature of Ω and the geodesic curvature of $\partial\Omega = \sqcup \partial\Omega_k$, respectively), we get

$$A(\Omega) = 2\pi\,(2g - 2 + N_0). \tag{4.37}$$

From this, and specializing (4.35) to the irreps of SU(2), we eventually get

$$Z^{Y-M}\left(U_1, \ldots, U_{\mathrm{No}}; A(\Omega)\right) = \sum_{j \in \left\{\frac{1}{2}\mathbb{Z}_+\right\}} \left(\frac{e^{-2\pi\, j(j+1)\beta_{\mathrm{YM}}^2}}{2j+1}\right)^{(2g-2+N_0)} \prod_{i=1}^{N_0} \frac{\sin\left[(2j+1)\frac{\Theta(i)}{2}\right]}{\sin\frac{\Theta(i)}{2}} \, . \tag{4.38}$$

In line with the preceding analysis, it is natural to consider the full partition (actually a N_0-points) function

$$\begin{aligned} Z^{open}_{N_0,g,\beta^2_{\mathrm{YM}}}((L_1, W^\Lambda(\infty,k), \Theta(k)); \ldots) &\doteq \frac{1}{2^{2|\alpha|}(3g-3+N_0-|\alpha|)!} \times \\ &\times \sum_{\substack{(\alpha_i)\in(\mathbb{Z}_{\geq 0})^{N_0} \\ |\alpha| \leq 3g-3+N_0}} \sum_{j \in \left\{\frac{1}{2}\mathbb{Z}_+\right\}} \left(\frac{e^{-2\pi\, j(j+1)\beta_{\mathrm{YM}}^2}}{2j+1}\right)^{(2g-2+N_0)} \times \\ &\times \int_{\overline{\mathcal{M}}_{g,N_0}} \prod_{i=1}^{N_0} \frac{L_i^{2\alpha_i} \left\|\xi_B(\infty)\xi^B(i)\right\|^{2\alpha_i}}{\alpha_i!} \frac{\sin\left[(2j+1)\frac{\Theta(i)}{2}\right]}{\sin\frac{\Theta(i)}{2}} \psi_i^{\alpha_i}\, \omega_{WP}^{3g-3+N_0-|\alpha|}, \end{aligned} \tag{4.39}$$

and discuss under what conditions it yields a duality of the form (4.31). The analysis of this problem is quite more complicated than the one discussed so far. It goes far beyond the pure kinematical aspects analized here since it involves an in depth analysis of the boundary conformal field theory associated with the modular dynamics of the SU(2) gauge field on the bordered surface Ω. It will be presented in a companion paper which is in preparation. However, already at this stage it is clear that we are calling into play also the moduli spaces of stable bundles, in particular the variety $Hom(\pi_1(\Omega), \mathrm{SU}(2))/\,\mathrm{SU}(2)$ of representations, up to conjugacy, of the fundamental group of the bordered surface Ω in SU(2) such that the monodromy around $\partial\Omega_k$ lies in the conjugacy class of $\partial\Omega_k$. It is well known that such a representation variety can be identified with the moduli space of semi-stable holomorphic rank 2 vector bundles over Ω. In such a framework, fixing the conjugacy class of the monodromy around a boundary component $\partial\Omega_k$ plays the same role played by the datum of the length of the geodesic boundary of $\partial\Omega_k$ in the case of the moduli space $\mathcal{M}_{g,N_0}(L)$. Thus, one is expecting that also intersection theory over $Hom(\pi_1(\Omega), \mathrm{SU}(2))/\,\mathrm{SU}(2)$ plays a role in establishing open/closed duality, (in her analysis of the volume of $\mathcal{M}_{g,N_0}(L)$, Mirzhakani draws similar conclusions). Moreover, since the analysis of the geometry of $Hom(\pi_1(\Omega), \mathrm{SU}(2))/\,\mathrm{SU}(2)$ and of boundary conformal field theory for SU(2) is intimately connected with Chern-Simons theory, one may wonder if there is an explicit geometrical counterpart of this, analogous to the simple open/closed duality between hyperbolic bordered surfaces and random Regge triangulations. Quite remarkably, this is indeed the case, and we conclude our kinematical analysis of open/closed string duality by presenting, in the next section, the geometrical aspects involved in activating Chern-Simons theory.

5. Connection with hyperbolic 3-manifold

The connection between twistorially decorated Regge surfaces and hyperbolic surfaces with boundaries can be naturally extended to three-dimensional hyperbolic cone-manifolds. Recall that to the twistor field $\sigma^0(k) \longmapsto W^{\Lambda}(k,\infty)$ on $|T_l| \to M$ we can associate either the marked horosphere (Σ_k, z_k) or, equivalently, the (unique) geodesic $\gamma(k,\infty)$ in $\mathbb{H}^{3,+}_{\rm up}$ connecting the vertex $v^0(k)$ with the vertex at ∞ of an ideal tetrahedron $\sigma^3_{\rm hyp}(\infty,k,h,j)$ in $\mathbb{H}^{3,+}_{\rm up}$. In particular, to any two adjacent triangles sharing a common edge, say $\sigma^2(k,h,j)$ and $\sigma^2(k,j,l)$, correspond pairwise adjacent tetrahedra, $\sigma^3_{\rm hyp}(\infty,k,h,j)$ and $\sigma^3_{\rm hyp}(\infty,k,j,l)$, that can be glued along the isometric faces $\sigma^2_{\rm hyp}(j,k,\infty)$ and $\sigma^2_{\rm hyp}(\infty,k,j)$. Each face-pairing is realized by an isometry of $\mathbb{H}^{3,+}_{\rm up}$

$$f_{jk} : \sigma^2_{\rm hyp}(j,k,\infty) \longrightarrow \sigma^2_{\rm hyp}(\infty,k,j) \tag{5.1}$$

which reverses orientation (so as to have orientability of the resulting complex). In this way, by pairwise glueing the $q(k)$ ideal tetrahedra $\left\{\sigma^3_{\rm hyp}(\infty,k,h_\alpha,h_{\alpha+1})\right\}$, associated with the corresponding Euclidean triangles $\sigma^2(k,h_\alpha,h_{\alpha+1})$, we generate a polytope

$$P^3(k) \doteq \coprod_{\alpha=1}^{q(k)} \sigma^3_{\rm hyp}(\infty,k,h_\alpha,h_{\alpha+1}) \Big/ \{f_{h_\alpha k}\} \tag{5.2}$$

with a conical singularity along the core geodesic $\gamma(k,\infty)$. Explicitly, let us denote by $\psi_{\cdot,\cdot}$ the dihedral angles associated with the edges $\sigma^1_{\rm hyp}(\cdot,\cdot)$ of this polytope. From the relations between the dihedral angles of each hyperbolic tetrahedron $\sigma^3_{\rm hyp}(\infty,k,h,j)$ and the vertex angles of the corresponding Euclidean triangle $\sigma^2(k,h,j)$ it easily follows that

$$\begin{aligned} \psi_{\infty,h} &= \theta_{hj\,k} + \theta_{k\,jl}\ , \\ \psi_{k\,j} &= \theta_{khj} + \theta_{klj}\ , \\ \psi_{hj} &= \theta_{hkj}\ , \\ \psi_{\infty,k} &= \sum_{\alpha=1}^{q(k)} \theta_{\alpha,k,\alpha+1} = \Theta(k). \end{aligned} \tag{5.3}$$

Note in particular that the conical defect $\Theta(k)$ at the vertex $\sigma^0(k) \in Star[\sigma^0(k)]$ propagates as a conical defect along the core geodesic $\gamma(k,\infty)$ of $\mathbb{H}^{3,+}_{\rm up}$. It follows that $P^3(k)$ has a non-complete hyperbolic metric and that the singularity on $\gamma(k,\infty)$ is conical with angle $\Theta(k)$. In order to endow $P^3(k)$ with a hyperbolic structure, let $\widetilde{P}_\gamma(k)$ denote the universal cover in $\mathbb{H}^{3,+}_{\rm up}$ of $P^3(k)$, with the core geodesic $\gamma(k,\infty)$ removed. $\widetilde{P}_\gamma(k)$ carries a natural hyperbolic structure and the holonomy representation of its fundamental group, $\pi_1(\widetilde{P}_\gamma(k)) = \mathbb{Z}$, is generated by an isometry of $\widetilde{P}_\gamma(k) \subset \mathbb{H}^{3,+}_{\rm up}$ of the form

$$\begin{aligned} \rho(k) : \pi_1(\widetilde{P}_\gamma(k)) &\longrightarrow Isom\left(\mathbb{H}^{3,+}_{\rm up}\right) \\ (c_s, s) &\longmapsto \left[a(k)\begin{pmatrix} e^{i\,\phi(s)} & 0 \\ 0 & e^{-i\,\phi(s)} \end{pmatrix}, s\right], \end{aligned} \tag{5.4}$$

where $a(k) > 1$ and $s \mapsto c_s$, $0 \leq s < \infty$ is closed curve winding around the link of $\sigma^0(k)$ in $Star\left[\sigma^0(k)\right]$ with $\phi(s = 2\pi) = \Theta(k)$. Since $a(k) > 1$, the isometry is hyperbolic (fixing the point $v^0(k)$ and ∞ in $\mathbb{H}^{3,+}_{\mathrm{up}}$). For simplicity, let us identify $v^0(k)$ with the origin of $\mathbb{H}^{3,+}_{\mathrm{up}}$. The horosphere Σ_k intersects $\sqcup\sigma^3_{\mathrm{hyp}}(\infty, k, h_\alpha, h_{\alpha+1})$ along a sequence of offset horocycle segments $\left\{\digamma_k^{h_\alpha}\right\}$ such that

$$d_{\mathbb{H}^3}(\digamma_k^{h_1}, \widehat{\digamma}_k^{h_{q(k)}}) = \left|\ln \frac{\sum_{\alpha=1}^{q(k)} \theta_{\alpha+1,k,\alpha}}{2\pi}\right|. \tag{5.5}$$

Similarly the concentric horosphere ${}^*\Sigma_k$ defined by $z = a(k)\, z_k$, $a(k) \geq 1$, intersects the $\sqcup\sigma^3_{\mathrm{hyp}}(\infty, k, h_\alpha, h_{\alpha+1})$ along a sequence of horocycle segments $\left\{{}^*\digamma_k^{h_\alpha}\right\}$ such that

$$d_{\mathbb{H}^3}({}^*\digamma_k^{h_1}, {}^*\widehat{\digamma}_k^{h_{q(k)}}) = \left|\ln \frac{\Theta(k)}{2\pi}\right|.$$

Let us consider the rectangular parallelepiped labeled by the segments $\left(\digamma_k^{h_1}, \widehat{\digamma}_k^{h_{q(k)}}; {}^*\digamma_k^{h_1}, {}^*\widehat{\digamma}_k^{h_{q(k)}}\right)$. A straightforward application of (3.10) provides the following relations between the (hyperbolic) lengths of the sides of this parallelepiped

$$\begin{aligned} \left|{}^*\widehat{\digamma}_k^{h_{q(k)}}\right| &= e^{-d_{\mathbb{H}^3}(\Sigma_k, {}^*\Sigma_k)} \left|\widehat{\digamma}_k^{h_{q(k)}}\right|, \\ \left|{}^*\digamma_k^{h_1}\right| &= e^{-d_{\mathbb{H}^3}(\Sigma_k, {}^*\Sigma_k)} \left|\digamma_k^{h_1}\right|, \end{aligned} \tag{5.6}$$

where $|\ldots|$ denotes the length of the corresponding horocycle segment. Since

$$d_{\mathbb{H}^3}(\Sigma_k, {}^*\Sigma_k) = 2\, \tanh^{-1} \frac{a(k) - 1}{a(k) + 1} = \ln\, a(k), \tag{5.7}$$

we get

$$\left|{}^*\widehat{\digamma}_k^{h_{q(k)}}\right| = a(k)^{-1} \left|\widehat{\digamma}_k^{h_{q(k)}}\right|, \qquad \left|{}^*\digamma_k^{h_1}\right| = a(k)^{-1} \left|\digamma_k^{h_1}\right|. \tag{5.8}$$

Moreover, from (4.6) we have

$$\left|\widehat{\digamma}_k^{h_{q(k)}}\right| = e^{\left|\ln \frac{\Theta(k)}{2\pi}\right|} \left|\digamma_k^{h_1}\right|, \qquad \left|{}^*\widehat{\digamma}_k^{h_{q(k)}}\right| = e^{\left|\ln \frac{\Theta(k)}{2\pi}\right|} \left|{}^*\digamma_k^{h_1}\right|. \tag{5.9}$$

By comparing these expressions, it follows that we can match the length of horocycle segment ${}^*\widehat{\digamma}_k^{h_{q(k)}}$ with the length of the segment $\digamma_k^{h_1}$ if we choose the parameter $a(k)$ according to

$$a(k) = e^{\left|\ln \frac{\Theta(k)}{2\pi}\right|}. \tag{5.10}$$

Such a matching condition allows, under the action of $\rho(k)\left(\pi_1\left(\widetilde{P}_\gamma(k)\right)\right)$, an (offset) identification between opposite faces of $\left(\digamma_k^{h_1}, \widehat{\digamma}_k^{h_{q(k)}}; {}^*\digamma_k^{h_1}, {}^*\widehat{\digamma}_k^{h_{q(k)}}\right)$, and consequently we can choose this rectangular parallelepiped as a fundamental domain for the action of the holonomy representation $\rho(k)$. The resulting developing map describes $\widetilde{P}_\gamma(k)$ as an incomplete manifold and $P^3_{\mathrm{hyp}}(k) \doteq \widetilde{P}_\gamma(k) \setminus \rho(k)$ is topologically equivalent to a solid torus $\mathbb{S}^1 \times B^2$, (B^2 being the meridianal 2-dimensional disc) with the central geodesic missing. Note that such a geodesic can be naturally identified with the geodesic boundary component $\partial\Omega_k$ of the open hyperbolic surface Ω. In order to get an intuitive picture of what happens, observe that the identification polytope $P^3(k)$, cut by the horosphere Σ_∞, is topologically a solid cylinder sliced by the faces of the component tetrahedra. If we remove a tube of small (infinitesimal) width around the central geodesic $\gamma(k,\infty)$ we get a topological solid torus sliced into parallelepipeds, with a thin and long tubular hole associated with the removed geodesic. The isometry (5.4) twists up this solid torus with a shearing motion, like a 3-dimensional photographic diaphragm. Adjacent parallelepipeds slide one over the other tilting up, while the central tube correspondingly winds up accumulating towards an horizontal $\mathbb{S}^1$.

5.1 Hyperbolic volume

We can formally extend this geometric analysis to the whole Regge triangulation $|T_l| \to M$ by forming the support space (for a compatible hyperbolic structure)

$$V \doteq \coprod_{\sigma^2_{\rm hyp}(l,m,\infty)}^{N_1(T)} \sigma^3_{\rm hyp}(\infty,k,h,j) \Big/ \{f_{lm}\}\,, \tag{5.11}$$

(the number of hyperbolic faces to be paired is equal to the number $N_1(T)$ of edges in $|T_l| \to M$). Note that the link of the vertex at ∞ in V is

$$link\ [\infty] \doteq \bigcup_{\sigma^1_{\rm hyp}(l,m)}^{N_1(T)} \sigma^2_{\rm hyp}(k,h,j), \tag{5.12}$$

where the glueing along the edges $\left\{\sigma^1_{\rm hyp}(l,m)\right\}$ is modelled after the Regge surface $|T_l| \to M$. If this latter has genus g, then from the Euler and Dehn-Sommerville relations

$$\begin{aligned} N_0(T) - N_1(T) + N_2(T) &= 2-2g, \\ 2N_1(T) &= 3N_2(T), \end{aligned} \tag{5.13}$$

we get that the support space V has

$$N_2(T) = 2N_0(T) + 4g - 4 \geq N_0(T) + g \tag{5.14}$$

ideal tetrahedra with $N_0(T)$ vertices associated with its boundary components ∂V. As we have seen in section 4, the edge-glueing of $\{\sigma^2_{\rm hyp}(k,h,j)\}$ gives rise to an incomplete hyperbolic surface and consequently also V cannot support, as it stands, a complete hyperbolic structure. To take care of this, we start by removing from V an open (horospherical) neighborhood of the vertices. In this way, each tetrahedron $\sigma^3_{\rm hyp}(\infty,k,h,j)$ becomes a octahedron with four (Euclidean) triangular faces (in the same similarity class which defines the given tetrahedron), and four (hyperbolic) exagonal faces. Note that the boundary of the removed open neighborhood of ∞ is triangulated by Euclidean triangles and it reproduces $|T_l| \to M$. Note also that the removed neighborhoods cut out an open disk D_k

around each vertex $v^0(k)$ in ∂V. Next, we remove from V also an open neighborhood of the geodesics $\{\gamma(k,\infty)\}_{k=1}^{N_0(T)}$. In this way we get from the support space V a handlebody H_V. Topologically, H_V is $[0,1]\times\Omega$, where Ω is the surface with boundary $(\sqcup\partial\Omega_k)$ associated with the hyperbolic completion of $\sqcup\sigma^2_{\text{hyp}}(k,h,j)$. The handlebody H_V plays here the role of the polytope $\widetilde{P}_\gamma(k)$ introduced in connection with the support space (5.2). By identifying the bottom $\Omega_0\simeq\partial H_V|_0$ and top $\Omega_1\simeq\partial H_V|_1$ copies of the surface Ω by means of the appropriate orientation reversing boundary homeomorphism $h:\partial H_V|_0\to\partial H_V|_1$, with $h(\partial\Omega_k|_0)=-\partial\Omega_k|_1$, we get the support space

$$V\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right)\setminus K\doteq H_V\setminus\sim^h \tag{5.15}$$

$(V\setminus K$, for notational ease), where K is the knot-link generated in H_V by the action of the identification homeomorphism h on the boundaries connecting the tubes associated with the removed core geodesics $\{\gamma(k,\infty)\}_{k=1}^{N_0(T)}$. It is not yet obvious that $V\setminus K$ admits a complete hyperbolic structure. First, we have been rather cavalier on the delicate issue concerning orientation in glueing the ideal tetrahedra, (for semi-simplicial triangulations problems connected with orientability of the hyperbolic complexes obtained upon face-identifications can be rather serious and we may end up in a ideal triangulation which may actually not define a manifold). Moreover, around the removed geodesics $\{\gamma(k,\infty)\}_{k=1}^{N_0(T)}$ the geometry is conical, and in order to establish completeness for the hyperbolic structure we have to discuss how hyperbolic Dehn filling can be extended to cone manifolds. These are delicate issues which, to the best of our knowledge, do not have answers that can be easily given in general terms. The interested reader may wish to consult the remarkable papers [39 – 41] where particular cases are thoroughly discussed. Notwithstanding the technical difficulties in characterizing complete hyperbolic structures on $V\setminus K$, their existence, when established, implies a number of important consequences which bear relevance to our analysis.

First of all, if the support space $V\setminus K$ generated by $|T_l|\to M$, is indeed a three-dimensional hyperbolic manifold $V_{\text{hyp}}(\{\Theta(k)\}\setminus K$, then we can easily compute its hyperbolic volume in terms of the conical angles $(\{\Theta(k)\}_{k=1}^{N_0(T)})$. As a matter of fact we can associate to any triangle $\sigma^2(k,h,j)$ of $|T_l|\to M$ the volume $Vol[\sigma^3_{\text{hyp}}]$ of the corresponding ideal tetrahedron σ^3_{hyp}. According to Milnor's formula, (see e.g. [28], for a very informative analysis), such a volume can be expressed in terms of the Lobachevsky functions $\mathcal{L}(\theta_{jk\,h})$, $\mathcal{L}(\theta_{khj})$, and $\mathcal{L}(\theta_{hjk})$ of the respective vertex angles of $\sigma^2(k,h,j)$, where

$$\mathcal{L}(\theta_{jk\,h})\doteq-\int_0^{\theta_{jk\,h}}\ln\,|2\sin\,x|\;dx. \tag{5.16}$$

In our setting, this translates into the mapping

$$\sigma^2(k,h,j)\longmapsto Vol\left[\sigma^3_{\text{hyp}}(\infty,k,h,j)\right]=\mathcal{L}(\theta_{j\;kh})+\mathcal{L}(\theta_{khj})+\mathcal{L}(\theta_{hj\,k}), \tag{5.17}$$

which is well-defined since, due to the symmetries of the dihedral angles of $\sigma^3_{\text{hyp}}(\infty,k,h,j)$, the valutation of $Vol\left[\sigma^3_{\text{hyp}}(\infty,k,h,j)\right]$ is independent from which vertex of the tetrahedron is actually mapped to ∞, (see [28], prop. C.2.8). Thus, we can compute the volume of the three-dimensional hyperbolic manifold $V_{\rm hyp} \setminus K$ as

$$Vol\left[V_{\rm hyp}\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right)\setminus K\right] = \sum_{\{\sigma^2(k,h_\alpha,h_{\alpha+1})\}}^{N_2(T)} \left[\mathcal{L}(\theta_{\alpha+1\,,\,k,\,\alpha}) + \mathcal{L}(\theta_{k,\,\alpha,\,\alpha+1}) + \mathcal{L}(\theta_{\alpha,\,\alpha+1,\,k})\right], \tag{5.18}$$

where the summation extends over all triangles $\sigma^2(k, h_\alpha, h_{\alpha+1})$ in the Regge triangulated surface $|T_l| \to M$. Equivalently, in terms of the complex moduli $\zeta_{\alpha+1\,,\,k,\,\alpha}$ of the triangles $\sigma^2(k, h_\alpha, h_{\alpha+1})$, we get

$$Vol\left[V_{\rm hyp}\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right)\setminus K\right] = \sum_{\{\sigma^2(k,h_\alpha,h_{\alpha+1})\}}^{N_2(T)} \left[\mathcal{L}(\arg\zeta_{\alpha+1\,,\,k,\,\alpha}) + \mathcal{L}(\arg\zeta_{k,\,\alpha,\,\alpha+1}) + \mathcal{L}(\arg\zeta_{\alpha,\,\alpha+1,\,k})\right]. \tag{5.19}$$

It is worthwhile to remark that if one computes the Hessian of $Vol\,[V_{\rm hyp}]$ with respect the angular variables $\{\theta_{\alpha+1\,,\,k,\,\alpha}\}$ of the generic triangle $\sigma^2(k, h_\alpha, h_{\alpha+1})$ one gets

$$\begin{aligned}
H_{k,\,k} &\doteq \frac{\partial^2}{\partial\,\theta_{\alpha+1\,,\,k,\,\alpha}{}^2}\,Vol\left[V_{\rm hyp}\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right)\right] = -\cot\,\theta_{\alpha+1\,,\,k,\,\alpha} = \\
&= \frac{l^2(h_{\alpha+1},k) + l^2(k,h_\alpha) - l^2(h_\alpha,h_{\alpha+1})}{4\,\Delta(\alpha+1\,,\,k,\,\alpha)}\,, \\
H_{\alpha,\,\alpha} &\doteq \frac{\partial^2}{\partial\,\theta_{k,\,\alpha,\,\alpha+1}{}^2}\,Vol\left[V_{\rm hyp}\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right)\right] = -\cot\,\theta_{k,\,\alpha,\,\alpha+1} = \\
&= \frac{l^2(k,h_\alpha) + l^2(h_\alpha,h_{\alpha+1}) - l^2(h_{\alpha+1},k)}{4\,\Delta(k,\alpha,\;\alpha+1)}\,, \\
H_{\alpha+1,\,\alpha+1} &\doteq \frac{\partial^2}{\partial\,\theta_{\alpha,\,\alpha+1,\,k}{}^2}\,Vol\left[V_{\rm hyp}\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right)\right] = -\cot\,\theta_{\alpha,\,\alpha+1,\,k} = \\
&= \frac{l^2(h_\alpha,h_{\alpha+1}) + l^2(h_{\alpha+1},k) - l^2(k,h_\alpha)}{4\,\Delta(\alpha,\;\alpha+1,k)},
\end{aligned} \tag{5.20}$$

where $\Delta \doteq \Delta(\alpha+1\,,\,k,\,\alpha)$ denotes, up to cyclic permutation, the Euclidean area of the triangle $\sigma^2(k, h_\alpha, h_{\alpha+1})$, (see paragraph 2.1). From (5.20) we get

$$\begin{aligned}
l^2(h_{\alpha+1},k) &= 2\,\Delta(H_{\alpha+1,\,\alpha+1} + H_{k,\,k}), \\
l^2(k,h_\alpha) &= 2\,\Delta\,(H_{k,\,k} + H_{\alpha,\,\alpha})\,, \\
l^2(h_\alpha,h_{\alpha+1}) &= 2\,\Delta(H_{\alpha,\,\alpha} + H_{\alpha+1,\,\alpha+1}),
\end{aligned} \tag{5.21}$$

which provide sign conditions on H_{lm}. Actually, it is relatively easy ([27]) to show that the restriction of the Hessian of $Vol\,[V_{\rm hyp} \setminus K]$ to the local Euclidean structure on each $\sigma^2(k, h_\alpha, h_{\alpha+1})$ is negative-definite. This latter remark implies that (minus) the Hessian of the hyperbolic volume can be used as a natural quadratic form on the space of deformations of the Euclidean structures associated with random Regge triangulations and which naturally pairs with the Weil-Petersson measure (4.19) on moduli space. It is also clear that formally the hyperbolic volume (5.18) does not require the existence of a complete hyperbolic structure on the support space $V \setminus K$, and we may well associate the function (5.18) to $V \setminus K$. However, the existence of a complete hyperbolic structure implies that such a volume function is a topological invariant by Mostow rigidity. Moreover, one can formulate the so-called volume conjecture (R. Kashaev and H. and J. Murakami) [17–19], (and [42] for a review), which, in our setting, may be phrased by stating that if K is not a split link and $J_n(K;t)$ is its colored Jones polynomial associated with the n-dimensional irreducible representation of $sl_2(\mathbb{C})$, then

$$2\pi \lim_{n\to\infty} \frac{\ln \left|J_n(K; \exp\left[\frac{2\pi i}{n}\right])\right|}{n} = Vol\left[V_{\text{hyp}}\left(\{\Theta(k)\}_{k=1}^{N_0(T)}\right) \setminus K\right] \tag{5.22}$$

(in the standard formulation of the volume conjecture the role of the support space $V(\{\Theta(k)\}_{k=1}^{N_0(T)})$ is played by $\mathbb{S}^3$, and one assumes that the complement $\mathbb{S}^3 \backslash K$ of the link K admits a (complete) hyperbolic structure). $J_n(K;t)$ is defined through the n-dimensional irreducible representations of the quantum group $U_q(sl(2,\mathbb{C}))$. For some hyperbolic knots in $\mathbb{S}^3$, in particular for the figure eight knot [20] (and for torus links, which are non-hyperbolic and yield 0 on the right member of (5.22)), the conjecture has been proved, (see also [43] for a deep analysis). This connection between knot polynomials and hyperbolic volume has been actually promoted to be part of a more general conjecture [21] relating the asymptotics of the colored Jones polynomials to the Chern-Simons invariant

$$2\pi\, i \cdot \lim_{n\to\infty} \frac{\ln\, J_n(K; \exp\left[\frac{2\pi i}{n}\right])}{n} = CS\, [V_{\text{hyp}} \diagup K] + i\, Vol\, [V_{\text{hyp}} \diagup K] \tag{5.23}$$

and

$$\lim_{n\to\infty} \frac{J_{n+1}(K; \exp\left[\frac{2\pi i}{n}\right])}{J_n(K; \exp\left[\frac{2\pi i}{n}\right])} = \exp\left(\frac{1}{2\pi i}\left(CS\, [V_{\text{hyp}} \diagup K] + i\, Vol\, [V_{\text{hyp}} \diagup K]\right)\right) \tag{5.24}$$

where again we have formally referred all quantities to $V_{\text{hyp}} \diagup K$, in particular $CS\, [V_{\text{hyp}} \diagup K]$ is the Chern-Simons invariant of the connection defined by the hyperbolic metric on $V_{\text{hyp}} \diagup K$. It should be clear that these statements have a status quite more conjectural then the original ones owing to the conical nature of $V_{\text{hyp}} \diagup K$, nonetheless they are reasonable in view of the holographic principle. Recall that a geometrical version of *classical* holography is familiar in hyperbolic geometry as the Ahlfors-Bers theorem which applies to hyperbolic manifolds V containing a compact subset determining a conformal structure on the boundary at ∞ of V. In such a case the geometry of V is uniquely determined by such induced conformal structure at ∞. It should be clear from its very set-up that our approach to closed/open duality is, geometricaly speaking, holographic in nature. Roughly speaking it is akin to a simplicial version of Ahlfors-Bers theorem, (for a serious analysis of this issue for conical hyperbolic manifolds see [44]). At this stage it is important to refer to the remarkable paper [45] which examines the connection between the volume of hyperbolic manifolds, the AdS/CFT correspondence and moduli space geometry. It would certainly interesting to analyze in depth the possible relation between their approach and our framework.

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References

[1] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[2] R. Gopakumar, *From free fields to AdS*, *Phys. Rev.* **D 70** (2004) 025009 [hep-th/0308184].

[3] R. Gopakumar, *From free fields to AdS. II*, *Phys. Rev.* **D 70** (2004) 025010 [hep-th/0402063].

[4] R. Gopakumar, *Free field theory as a string theory?*, *Comptes Rendus Physique* **5** (2004) 1111–1119 [hep-th/0409233].

[5] R. Gopakumar, *From free fields to AdS. III*, *Phys. Rev.* **D 72** (2005) 066008 [hep-th/0504229].

[6] J.R. David and R. Gopakumar, *From spacetime to worldsheet: four point correlators*, hep-th/0606078.

[7] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: liouville D-branes and the Kontsevich model*, *JHEP* **07** (2005) 053 [hep-th/0312196].

[8] O. Aharony, Z. Komargodski and S. S. Razamat, *On the worldsheet theories of strings dual to free large N gauge theories*, *JHEP* **05** (2006) 16 hep-th/0602226.

[9] E.T. Akhmedov, *Expansion in feynman graphs as simplicial string theory*, *JETP Lett.* **80** (2004) 218 [hep-th/0407018].

[10] K. Strebel, *Quadratic differentials*, Springer-Verlag (1984).

[11] M. Mulase, M. Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over $\overline{\mathbb{Q}}$*, *The Asian Journal of Mathematics* **2** (1998) 875 math-ph/9811024 v2.

[12] R.C. Penner *The decorated Teichmüller space of punctured surfaces, Commun. Math. Phys.* **113** (1987) 299.

[13] W.P. Thurston, *Three-Dimensional Geometry and Topology* Vol.I (edited by S. Levi) Princeton Univ. Press (1997).

[14] R. Kaufmann and R.C. Penner, *Closed/open string diagrammatics*, math.GT/0603485.

[15] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Preprint (2005).

[16] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli spaces of curves*, Preprint (2005).

[17] R.M. Kashaev, *A link invariant from quantum dilogarithm*, *Modern Phys. Lett. A* **10** (1995) 1409.

[18] R.M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, *Lett. Math. Phys.* **39** (1997) 269.

[19] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, *Acta Math.* **186** (2001) 85.

[20] H. Murakami, *Kashaev's invariant and the volume of a hyperbolic knot after Y. Yokota* from: *Physics and combinatorics 1999 (Nagoya)*, World Sci., River Edge, NJ (2001) 244.

JHEP12(2006)017

[21] H. Murakami, J. Murakami, M. Okamoto, T. Takata and Y. Yokota, *Kashaev's conjecture and the Chern-Simons invariants of knots and links*, *Experimental Mathematics* **11** (2002) 427.

[22] G. Mondello, *Triangulated Riemann surfaces with boundary and the Weil-Petersson Poisson structure*, math.DG/0610698.

[23] T. Regge and R.M. Williams, *Discrete structures in gravity*, *J. Math. Phys.* **41** (2000) 3964 [gr-qc/0012035];
H. Hamber, *Simplicial quantum gravity*, lecture notes published in *Gauge theories, critical phenomena and random systems*, Proceedings of the 1984 Les Houches Summer School, Session XLIII, (edited by K. Osterwalder and R. Stora, North Holland, 1986).

[24] J. Ambjörn, M. Carfora and A. Marzuoli, *The geometry of dynamical triangulations*, *Lecture Notes in Physics* **m50**, Springer Verlag (1997).

[25] J. Ambjörn, B. Durhuus and T. Jonsson, *Quantum Geometry*, *Cambridge Monograph on Mathematical Physics*, Cambridge Univ. Press (1997).

[26] M. Carfora and A. Marzuoli, *Conformal modes in simplicial quantum gravity and the Weil-Petersson volume of moduli space*, *Adv. Theor. Math. Phys.* **6** (2003) 357 [math-ph/0107028].

[27] I. Rivin, *Euclidean structures on simplicial surfaces and hyperbolic volume*, *Ann. of Math.* **139** (1994) 553.

[28] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, *Springer-Verlag*, New York (1992).

[29] P. Baird, *Riemannian twistors and Hermitiean structures on low-dimensional space forms*, *Jour. of Math. Phys.* **33** (1992) 3340.

[30] R.C. Penner, *Weil-Petersson volumes Jour. Diff. Geom.* **35** (1992) 559.

[31] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, *Surveys in diff. geom.* **1** (1991) 243.

[32] G. Schumacher and S. Trapani, *Estimates of Weil-Petersson volumes via effective divisors*, *Commun. Math. Phys.* **222** (2001) 1, math.AG/0005094 v2.

[33] S. Grushevsky, *Explicit upper bound for the Weil-Petersson volumes* Math. Ann. **321** 1 (2001), math.AG/0003217.

[34] P. Zograf, *Weil-Petersson volumes of moduli spaces of curves and curves and the genus expansion in two dimensional*, math.AG/9811026.

[35] Y.I. Manin and P. Zograf, *Invertible cohomological field theories and weil-petersson volumes*, math.ag/9902051.

[36] M. Kontsevich, *Intersection theory on th moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[37] E. Looijenga, *Intersection theory on Deligne-Mumford compactifications*, *Séminaire Bourbaki*, (1992-93), 768.

[38] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 184 [hep-th/9411210].

[39] C. Petronio and J. Porti *Negatively oriented ideal triangulations and a proof of Thuston's hyperbolic dehn filling theorem*, *Expo. Math.* **18** (2001) 1, math.GT/9901045.

[40] F. Costantino, R. Frigerio, B. Martelli and C. Petronio, *Triangulations of 3-manifolds, hyperbolic relative handlebodies, and Dehn filling*, math.GT/0402339.

[41] S. Moroianu, J-M. Schlenker *Hyperbolic cone-manifolds with singular infinity*, Preprint (2006).

[42] T. Ohtsuki (ed), *Problems on invariants of knots and 3-manifolds*, in: T. Kohno, T. Le, J. Murakami, J. Roberts and V Turaev (eds.), *Invariants of knots and 3-manifolds Geom. and topology monographs*, **4** (2002) 377.

[43] S. Baseilhac and R. Benedetti, *QHI, 3-manifolds scissors congruence classes and the volume conjecture* in: *Invariants of knots and 3-manifolds*, T. Ohtsuki et al. eds., *Geom. and topology monographs*, **4** (2002) 13, math.GT/0211053.

[44] S. Moroianu and J.-M. Schlenker, *Quasi-Fuchsian manifolds with particles* math.DG/0603441.

[45] K. Krasnov and J.-M. Schlenker, *On the renormalized volume of hyperbolic 3-manifolds*, math.DG/0607081.

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